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Incoherent imaging of a periodic point object using an aperture of black-white concentric annuli

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The incoherent imaging of a periodic point object consisting of equidistant unit impulses is considered in this study. An aperture composed of black-white concentric annuli is utilized. The intensity distribution in the imaging plane is calculated from the convolution product of the object intensity and the impulse response appropriate to the system. An expression of the modulation transfer function of the imaging system is deduced in matrix form. A computer program is constructed to compute the normalized autocorrelation function of some black-white apertures and compared with that obtained for a uniform circular aperture.

1. Introduction

The incoherent imaging system may be regarded as a linear shift-invariant system [1]. In this system, the light radiated by the object has a narrow temporal frequency spectrum, *i.e.*, if the light has a bandwidth $\Delta\lambda$ centered about the wavelength λ, we require it to fulfil the inequality $\Delta\lambda \ll \lambda$ but to be spatially incoherent. This differs from the coherent imaging system in which the light was assumed to be monochromatic and therefore perfectly coherent. Therefore, a coherent wave field is characterized by its ability to produce constructive and destructive interference when different portions of it are combined at the same location. The fundamental reason for this behaviour is that the realtionship between the phases at any two points in a coherent wave field is fixed in time. A monochromatic wave field would be coherent both spatially and temporally. But such a wave field cannot exist in the real world. On the other hand, no wave field in the real world is completely incoherent either. Thus, all the real wave fields exhibit some degree of partial coherence, and the notations of coherent light and incoherent light are contrived to simplify certain calculations. In the former case of incoherent imaging, the object may be either self-luminous or irradiated by some incident wave field; in either event, we shall consider the radiated light to be spatially incoherent. Hence, the diffracted imaging distribution is computed from the convolution product of the point spread function of the system and the irradiance of the geometrical image. In the other case of coherent imaging, the diffracted image is calculated from the modulus square of the convolution product of the amplitude spread function and the complex amplitude of the object.

The pioneer work of many scientists on the transfer function of the incoherent imaging systems with slit, circular, and annular apertures has been reported [2]–[6]. In most of these studies, incoherently illuminated sine wave objects have been taken as targets. Since these target objects are fabricated with a great difficulty, hence practical objects of rectangular or triangular wave shapes are suggested and realized easily. The frequency responses of the above-mentioned objects are investigated [7]–[10]. Studies on the evaluation of sine wave, rectangular and triangular wave responses of non-uniformly illuminated apertures have been made in [11]–[14]. In view of the above discussion, we made a theoretical study on a comb function as representing an object in the presence of a novel apodization of the aperture. In this case, the response of this object is calculated and the transfer function of the incoherent imaging system is computed for that aperture of B/W concentric annuli. The manipulation of this apodization is recommended for that hyper-resolving aperture. The analysis is followed by theoretical results and finally a conclusion is given.

2. Theoretical analysis

The intensity distribution in the imaging plane is given by the convolution of the object intensity distribution $E(r)$ with the impulse response $R(r')$ appropriate to the imaging system. In the former case of incoherent imaging, the intensity becomes

$$I(r') = \int_{-\infty}^{\infty} E(r')R(r'-r)dr \tag{1}$$

or symbolically as

$$I(r') = E(r') * R(r'), \tag{1a}$$

$*$ – symbol for convolution product, $r = (x,y)$ is the radial coordinate in the object plane, and $r' = (x',y')$ is the radial coordinate corresponding to the imaging plane.

Now, assume that the periodic point object represented by a comb function of equidistant unit impulses separated by r_0 is written as

$$E(r) = \sum_{n=-\infty}^{\infty} \delta(r - nr_0). \tag{2}$$

The Fourier series representation of such a wave can be shown to be given by

$$E(r) = \frac{1}{r_0} \sum_{n=-\infty}^{\infty} \exp(jn\omega_0 r) \tag{3}$$

where ω is the angular frequency given by $\omega_0 = 2\pi/r_0 = 2\pi\upsilon$, and υ is the number of repetitive elements per unit distance, so that the period is $r = 1/\upsilon$. This type of object can be generated artificially by drawing a series of a great number of circles of diminishing diameters separated by a distance r_0. Each Fourier component of the object $E(r)$ at frequencies $\omega = 0, \pm\omega_0, \ldots, \pm n\omega_0$ is modulated by a transfer function and therefore the imaging intensity distribution is given by

$$I(r') = \frac{1}{r_0} \sum_{n=-\infty}^{\infty} R(n\omega_0)\cos(n\omega_0 r') \tag{4}$$

where $R(n\omega)$ is the transfer function of the imaging system and is calculated from the Fourier transform of the spread function of the incoherent imaging system. This spread function or the intensity impulse response is calculated by taking the modulus square of the point spread function of a coherent imaging system, *i.e.*,

$$R(r') = |h(r')|^2 \tag{5}$$

where $h(r') = \mathrm{FT}\{A(w)\}$.

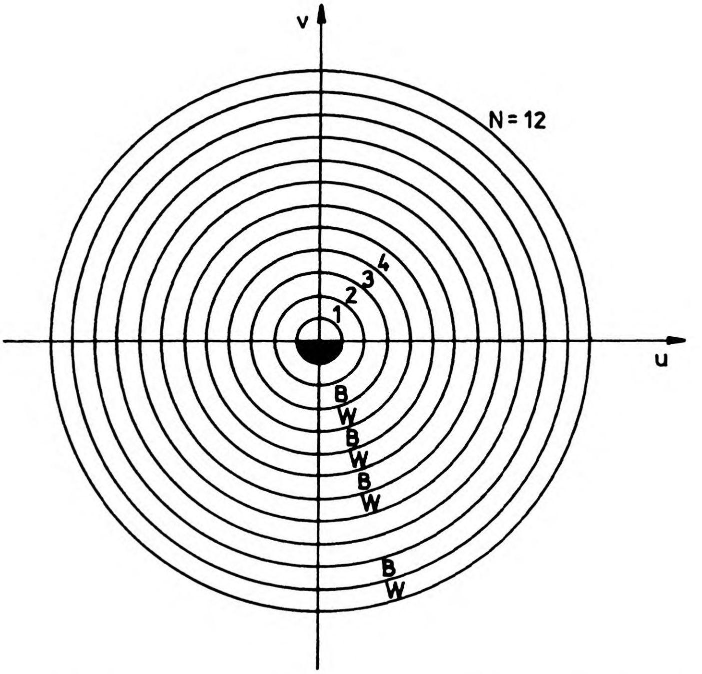


Fig. 1. General representation of black-white concentric annuli used as an aperture of a definite number of zones N, where the effective pupil is: $P_{\mathrm{eff}} = \sum_{m=1}^{M} (P_{2m} - P_{2m-1})$; $M = N/2$

It is known that the modulation transfer function of the incoherent imaging system is the autocorrelation function of the pupil aperture, *i.e.*,

$$C(w) = A(w) * A^{*}(w) \tag{6}$$

where $w = (u,v)$ is the radial coordinate in the plane of the aperture. It consists of a series of concentric annuli with black and transparent areas of equal width as shown in Fig. 1. This aperture is mathematically represented as follows:

$$A(w) = \sum_{i=1}^{N} \Delta A_i(w) \tag{7}$$

where $\Delta A_i(w) = A_i(w) - A_{i-1}(w)$ is the difference between any two successive circular apertures and is considered as an annulus of radial width $\Delta w = w' - w$, N is the total number of transparent and black zones constituting the whole aperture. The computations are made for a definite number of zones. The intensity impulse response (IMR) of the imaging system is obtained by operating the Fourier transform upon equation (7) to obtain this result

$$\text{IMR} = R(r') = \left[\sum_{i=1}^{N}\left\{\left(\frac{2J\mu' r'}{\mu' r'}\right) - \left(\frac{2J(\mu r')}{\mu r'}\right)\right\}\right]^2 \tag{8}$$

where: $\mu = 2\pi w/\lambda f$, $\mu' = 2\pi w'/\lambda f$.

From Equations (4) and (8), the intensity distribution of the incoherent imaging system is obtained. The object was described by a comb function and the aperture of the imaging system was a black-white concentric annuli. It is clear that at the fundamental frequency $\omega_0 = 0$, the intensity distribution becomes

$$I(r') = \frac{1}{r_0} R(\omega_0 = 0), \tag{9}$$

while for the first harmonic frequency, $\omega = \omega_0$,

$$I(r') = \frac{1}{r_0} R(2\pi/r_0)\cos(\omega_0 r'); \quad \omega_0 = 2\pi/r_0. \tag{10}$$

In this study, we confine ourselves to definite number of zones N considering that a series of black-white concentric annuli represent the aperture of the system. The convolution product of two apertures is calculated for $N = 2$ as follows:

$$C_2(w) = (P_2 - P_1) * (P_2 - P_1) = P_1 * P_1 + P_2 * P_2 - 2P_1 * P_2.$$

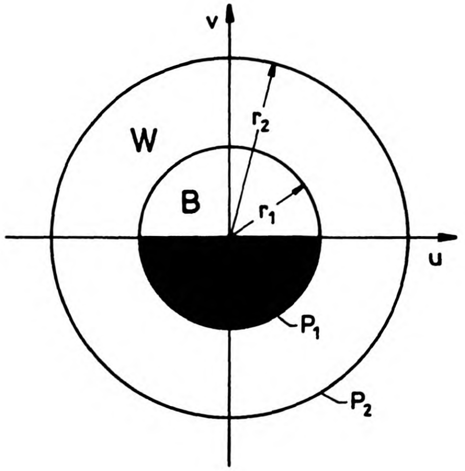
This transfer function can be represented in a matrix form as follows:

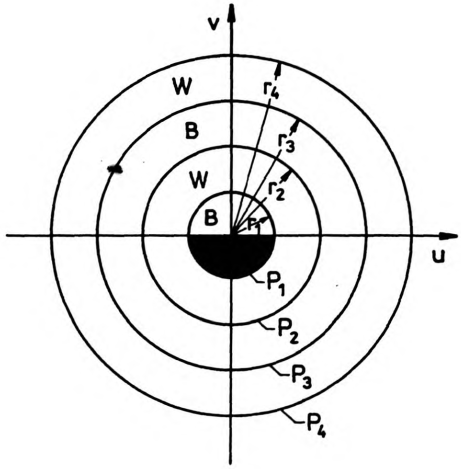
$$C_2(w) = \begin{bmatrix} P_1 & P_2 \\ -2P_2 & 0 \end{bmatrix} * \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}. \tag{11}$$

In a similar form, $C(w)$ is obtained in a matrix form ($N = 4$) as follows:

$$C_4(w) = \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \\ -2P_2 & -2P_3 & -2P_4 & 0 \\ 2P_3 & 2P_4 & 0 & 0 \\ -2P_4 & 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}. \tag{12}$$

The apertures used for calculating the transfer functions are represented as shown in Fig. 2 for $N = 12$, Fig. 3 for $N = 4$, and in Fig. 4 for $N = 6$. The radii corresponding to each aperture are represented in the figures.





▲ Fig. 3

Fig. 2. Black-white concentric annuli of $N = 2$. The aperture consists of two concentric circles. The inner zone is a black circle of radius $r_1 = 0.25$ and the outer zone is transparent of radius $r_2 = 0.5$, $P_{\text{eff}} = P_2 - P_1$

Fig. 3. Black-white concentric annuli of $N = 4$. The aperture consists of four zones, of which only two are transparent. The radii are: $r_1 = 0.25$, $r_2 = 0.5$, $r_3 = 0.75$, $r_4 = 1.0$ and the effective pupil is calculated as: $P_{\text{eff}} = (P_4 - P_3) + (P_2 - P_1)$

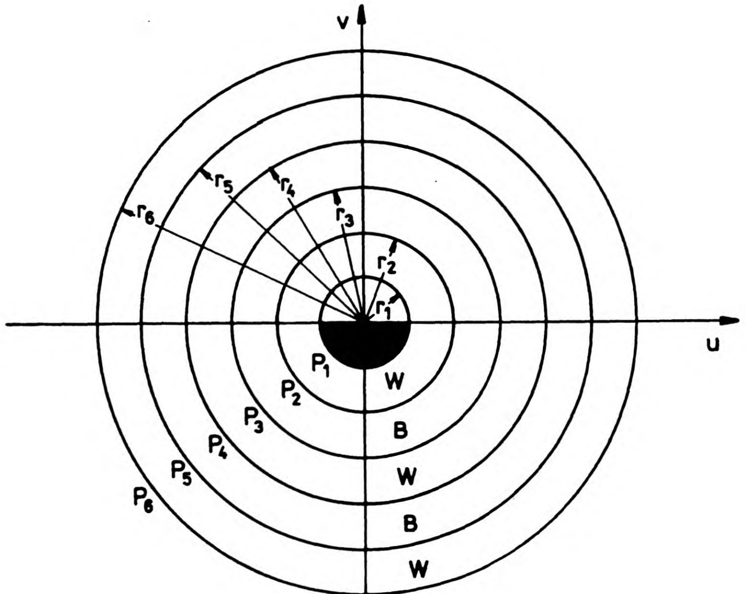


Fig. 4. Black-white concentric annuli of $N = 6$. The radii are: $r_1 = 0.25$, $r_2 = 0.5$, $r_3 = 0.75$, $r_4 = 1.0$, $r_5 = 1.25$, $r_6 = 1.5$ and the effective pupil is: $P_{\text{eff}} = (P_6 - P_5) + (P_4 - P_3) + (P_2 - P_1)$

For $N = 6$, we obtain this result for the fransfer function

$$C_6(w) = \begin{bmatrix} P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \\ -2P_2 & -2P_3 & -2P_4 & -2P_5 & -2P_6 & 0 \\ 2P_3 & 2P_4 & 2P_5 & 2P_6 & 0 & 0 \\ -2P_4 & -2P_5 & -2P_6 & 0 & 0 & 0 \\ 2P_5 & 2P_6 & 0 & 0 & 0 & 0 \\ -2P_6 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix} \tag{13}$$

In general, for N zones, the convolution matrix becomes

$$C_N(w) = \begin{bmatrix} P_1 & P_2 & P_3 & \cdots & \cdots & \cdots & P_N \\ -2P_2 & -2P_3 & \cdots & \cdots & \cdots & -2P_N & 0 \\ 2P_3 & 2P_4 & 2P_5 & \cdots & 2P_N & 0 & 0 \\ \cdots & -2P_5 & \cdots & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 2P_{N-1} & 2P_N & 0 & \cdots & \cdots & \cdots & 0 \\ -2P_N & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix} * \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ \cdots \\ \cdots \\ \cdots \\ P_N \end{bmatrix} \tag{14}$$

3. Theoretical results

It is known that the convolution of any two symmetric circular apertures is calculated as [1]

$$C_{ii}(w) = P_i * P_i = r_i^2\{\cos^{-1}(w/2r_i) - (w/2r_i)[1-(w/2r_i)^2]^{1/2}\} \tag{15}$$

where $r_i = r_1, r_2, \ldots, r_N$, N – total number of zones.

The convolution of any two different circular apertures i,j is calculated as [1]

$$C_{ij}(w) = P_i * P_j = r_i^2 - \cos^{-1}(\alpha) - \alpha(1-\alpha)^{1/2} + r_j^2\cos^{-1}(\beta) - \beta(1-\beta)^{1/2} \tag{16}$$

where: $\alpha = \{[w^2 + (r_i^2 - r_j^2)]/2wr_i\}$ and $\beta = \{[w^2 + (r_j^2 - r_i^2)]/2wr_j\}$.

By substituting the corresponding expressions from Eq. (15) and Eq. (16) in the convolution matrix $C(w)$ given by Eq. (14), the transfer function of the imaging system can be computed. A computer program in Fortran has been written to compute the convolution matrices corresponding to a definite number of zones N. The transfer function of a circular aperture is given for comparison with the results obtained. It is seen, referring to Eq. (4), that the spatial harmonics of the periodic impulse function are attenuated by the transfer function corresponding to its frequency, hence the image intensity distribution is obtained by adding these attenuated harmonics. The series given by Eq. (4) consists of a finite number of terms because there is a limiting value of ω beyond which the transfer function falls to zero for all larger values of ω, *i.e.*, the cut-off spatial frequency must satisfy the following condition: $\omega \leqslant \omega_c$, $\omega_c = r_0/\lambda f$, where f – focal length of the Fourier transform lens of the imaging system. The above condition is only valid in the case of uniform circular apertures. In the former case of B/W concentric annuli, the cut-off spatial frequency is dependent upon the results of computations obtained from the convolution matrix which is dependent upon the total number of utilized zones N. The computation of the convolution matrix for $N = 2$ has been performed, where the internal black circle has a radius $r_1 = 0.25$, while the external transparent circle has a radius $r_2 = 0.5$ Four different curves of convolution are drawn in Fig. 5 using Eq. (11). The cut-off spatial frequencies corresponding to the curves in Fig. 5 are located at $r_{c_{11}} = 0.5$, $r_{c_{22}} = 1.0$ and $r_{c_{12}} = 0.75$. The curve obtained for number of

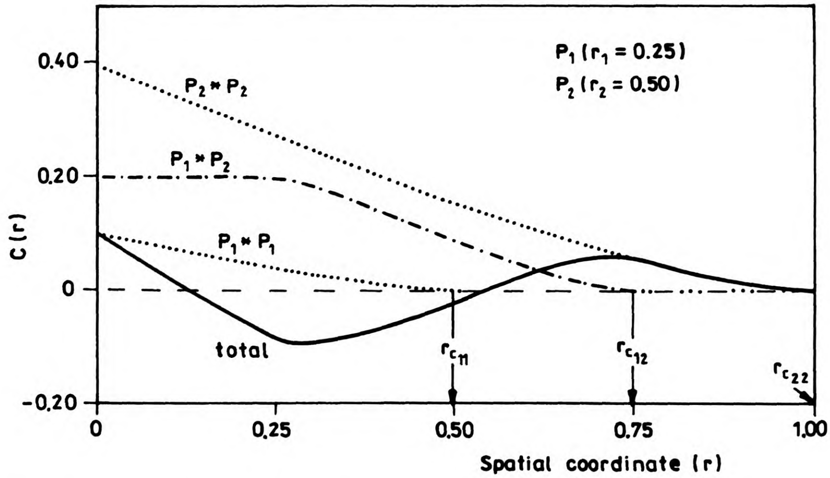


Fig. 5. Convolution product of two concentric annuli calculated as: $C(\text{total}) = P_1 * P_1 + P_2 * P_2 - 2(P_1 * P_2)$

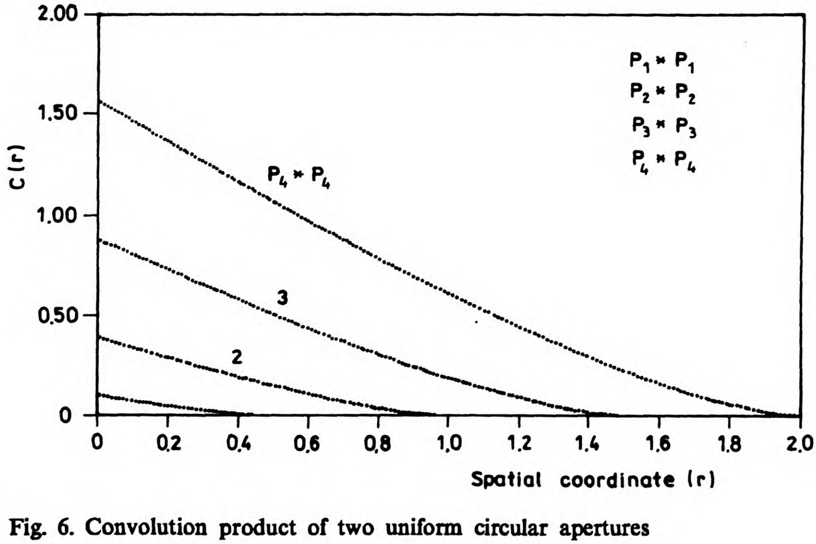


Fig. 6. Convolution product of two uniform circular apertures

zones $N = 2$ shows an oscillation between a minimum at $r = 0.27$ and a maximum at $r = 0.75$.

A set of four curves are given for comparison, corresponding to the convolution product of two symmetric circular apertures as shown in Fig. 6. The cut-off spatial frequencies are located at $r_{c_{11}} = 0.5$, $r_{c_{22}} = 1.0$, $r_{c_{33}} = 1.5$, and $r_{c_{44}} = 2.0$. These results are obtained easily using Eq. (15). Another set of six curves are obtained from Eq. (16) as shown in Fig. 7 representing the convolution product of two circular

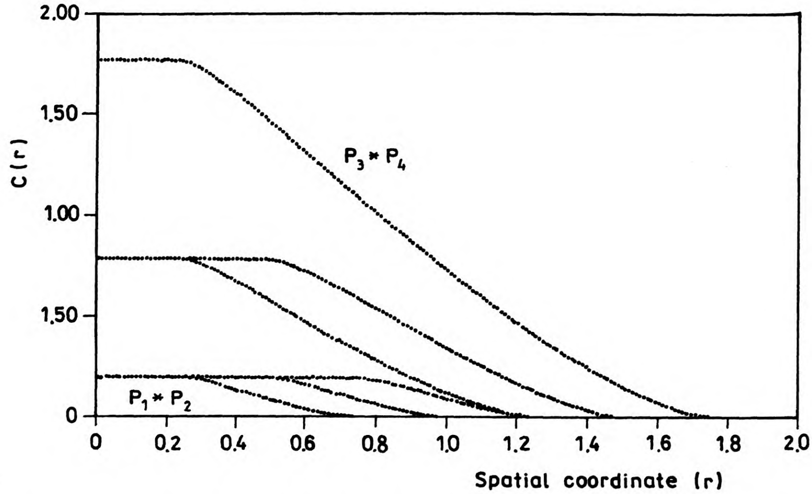


Fig. 7. Convolution product of $P_1 * P_2$, $P_1 * P_3$, $P_1 * P_4$, $P_2 * P_3$, $P_2 * P_4$ and $P_3 * P_4$, respectively

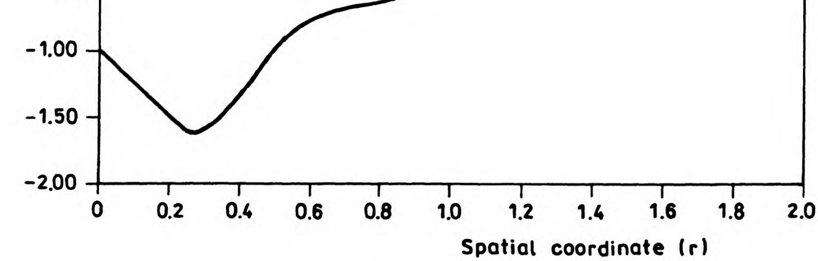


Fig. 8. Convolution product of two modulated apertures calculated as: $C(r) = P_1 * P_1 + P_2 * P_2 + P_3 * P_3 + P_4 * P_4 + 2P_1 * P_3 + P_2 * P_4 - 2;\ P_1 * P_2 + P_3 * P_4 + P_1 * P_4$ (* symbol for convolution product)

apertures of different radii. The cut-off is located at these values for the apertures: (1, 2) = 0.75, (1, 3) = 1.0, (1, 4) = 1.25, (2, 3) = 1.25, (2, 4) = 1.5, (3, 4) = 1.75. The resultant convolution matrix $C(w)$ is obtained from the summation of the above curves, which is calculated from Eq. (12) giving a curve shown in Fig. 8. That curve decreases until reaching a minimum value at $r = 0.25$ and then increases with a smooth fluctuation until reaching a maximum value at $r = 1.75$. Finally, the convolution matrix obtained for $N = 6$ using Eq. (13) is graphically constructed as shown in Fig. 9, giving a minimum value at $r = 0.25$ and then increases with a

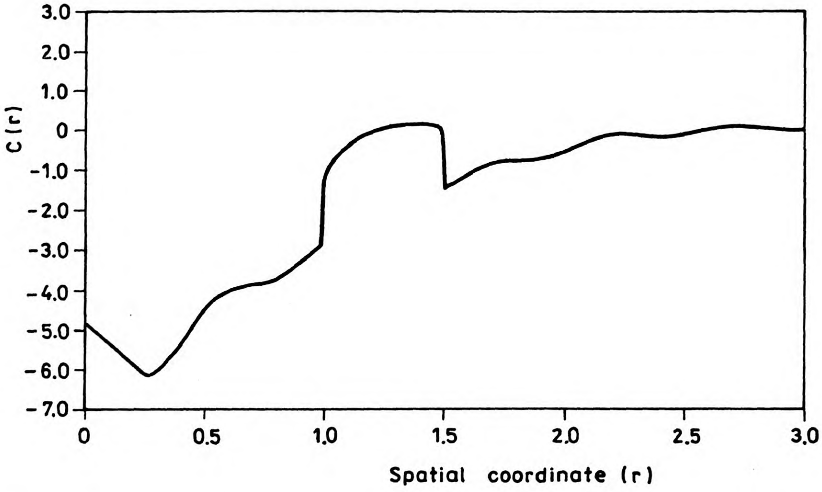


Fig. 9. Convolution product of two modulated apertures, each having black-white concentric circles (six zones are used)

small fluctuation until reaching the point $r = 1.0$. The curve increases sharply in a nearly rectangular shape in the region [1.0, 1.5] with a curvature in the envelope of the gate and then an abrupt decrease occurs at $r = 1.5$ and increases again with a slight variation in the range [1.5, 3.0].

4. Conclusion

Firstly, the intensity distribution in the imaging plane is obtained using the periodic impulse as a target represented by a comb function. The aperture of the imaging system is composed of a series of black-white concentric annuli.

Secondly, the transfer function of the incoherent imaging system is computed using the black-white concentric annuli as an aperture. The computed results of the convolution matrices which represent the transfer function are dependent upon the number of utilized zones. A general expression of the transfer function is obtained in a matrix form for black-white annuli and compared with the corresponding autocorrelation function valid in the case of uniform circular apertures. The former aperture of black-white annuli is considered as a hyper-resolving aperture, since it gives a wide transfer function for a sufficiently great number of zones N. This aperture is useful for imaging the microscopic objects giving better resolution than the clear circular apertures.

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