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Fractional Fourier transform in optical setups

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The fractional order Fourier transform was introduced into optics on the basis of light propagation through a piece of graded-index fiber of proper length. Lohmann reinvented the fractional Fourier transform operation based on the Wigner distribution function that can be applied not only for wave propagation in free space, but for passage through a lens system, too. In this paper, it is shown that optical implementation of the fractional Fourier transform of different orders requires optical systems of different (Fourier planes) configurations.

1. Introduction

It is well known that mathematical operation of the conventional Fourier transform can be optically implemented by lenses and plays a fundamental role in Fourier optics. In recent years it has been shown that optical implementation of the fractional Fourier transform exists as a more general operation [1]–[5]. Invented by NAMIAS [1], the fractional Fourier transform is well adapted to describing the operation of optical wave fronts, in particular through quadratic refractive index (GRIN) media, and through discrete refractive elements (*i.e.*, lenses) in optical setups of different configurations.

MENDLOVIC and OZAKTAS [2]–[4] were the first to introduce the fractional Fourier transform into optics on the basis of the fact that a graded-index fiber of proper length is capable of performing a Fourier transform of the input object. Then, LOHMANN [5] reinvented the fractional Fourier transform based on the Wigner distribution function, the horizontal and vertical shearing of which corresponds to wave propagation in free space and passage through a lens, respectively.

The Fourier transform of an object amplitude distribution determines a quantitative image of the frequency content of the object and is fundamental to the processing and analysis of images; it has certain properties which make it particularly versatile and easy to work with. In this paper, it is shown that optical implementation of fractional Fourier transform of different orders requires optical systems of different configurations, whereas conventional Fourier transform can be realized between the front and back focal planes of a lens as a particular case of the fractional Fourier transform.

2. Definitions of the fractional Fourier transform

It is often necessary or desirable to describe a function $f(x)$ that is absolutely integrable on the interval $(-\infty;\infty)$, and has a Fourier transform defined by

$$\tilde{F}(\xi,\eta) = \iint_{-\infty}^{\infty} f(x,y)\exp[-i2\pi(x\xi+y\eta)]\mathrm{d}x\mathrm{d}y,$$

or its inverse (1)

$$f(x,y) = \iint_{-\infty}^{\infty} \tilde{F}(\xi,\eta)\exp[i2\pi(x\xi+y\eta)]\mathrm{d}\xi\mathrm{d}\eta.$$

Such an operation called conventional (or classical) Fourier transform can be easily implemented by lenses and is widely applied in Fourier optics [6]. The fractional Fourier transform, as was mentioned, can also be implemented by optical imaging systems, and (because of different orders of this operation) could rather be called a generalized Fourier transform.

2.1. Quadratic graded-index medium

The fractional transform can be determined by the integer or by the noninteger orders p, and the p-th fractional order Fourier transform of a function $f(x,y)$ will be denoted as $\mathscr{F}^p\{f(x,y)\}$. For $p = 1$ we obtain the first order Fourier transform called the conventional Fourier transform.

First of all, the definition of optical fractional Fourier transform was based on the optical field during propagation along a quadratic graded-index medium [2], [4]. In such a medium propagation and focusing take place simultaneously, because of its refractive index profile given by

$$n^2(r) = n_1^2\left(1-\frac{n_2}{n_1}r^2\right) \tag{2}$$

where $r = \sqrt{x^2+y^2}$ is the radial distance from the optical axis, and n_1, n_2 are the graded-index medium parameters. It has been shown that a plane wave front incident normally at the input plane is focused at a distance proportional to the order p of fractional Fourier transform

$$z_p = \frac{\pi p}{2}\sqrt{\frac{n_1}{n_2}} = pL. \tag{3}$$

In other words, for Fourier transform of fractional order $p = 1$ the focal length of this medium $f = z_1 = L$. If the object function $u_O(x,y)$ is placed in the input plane of the GRIN medium, we obtain at its focal plane the first order Fourier transform of this function in the expression $F\{u_O(x,y)\}$ given by Eq. (1). Because of uniform graded-index medium in the optical axis direction, the fractional Fourier transform of the object function $u_O(x,y)$ is expressed at the plane the position of which is defined by Eq. (3). The self-modes of quadratic graded-index media are the

Hermite–Gaussian functions [7] which form a mutually orthogonal set, whereas the l-th and m-th member of this set is given by

$$\Psi_{lm}(x, y) = H_l\left(\frac{\sqrt{2}x}{\omega}\right)H_m\left(\frac{\sqrt{2}y}{\omega}\right)\exp\left(-\frac{x^2+y^2}{\omega^2}\right)$$

where H_l and H_m are the Hermite polynomials of orders l and m, respectively, and if the wave length is denoted by λ, the "spot size" is expressed as

$$\omega = \sqrt{\frac{\lambda}{\pi}}\left(\frac{1}{n_1 n_2}\right)^{1/4}.$$

Any two-dimensional function $f(x, y)$ can be expressed as a linear combination of the Hermite–Gaussian functions

$$f(x, y) = \sum_l\sum_m A_{lm}\Psi_{lm}(x, y) \tag{4}$$

where the coefficient

$$A_{lm} = \iint\limits_{-\infty}^{\infty}\frac{f(x, y)\Psi_{lm}(x, y)}{h_{lm}}\mathrm{d}x\mathrm{d}y,$$

and $h_{lm} = 2^{l+m}l!m!\dfrac{\pi\omega^2}{2}$. If the propagation constant of the l-th and m-th mode is described by

$$\beta_{lm} = \frac{2\pi n_1}{\lambda}\left[1-\frac{\lambda}{\pi n_1}\sqrt{\frac{n_2}{n_1}}(l+m+1)\right]^{1/2}, \tag{5}$$

then the p-th order fractional Fourier transform of $f(x, y)$ is defined as

$$\mathscr{F}^p\{f(x, y)\} = \sum_l\sum_m A_{lm}\Psi_{lm}(x, y)\exp(i\beta_{lm}z_p). \tag{6}$$

For $p = 1$ the distance of propagation $z_p = L$ in the graded-index medium provides the conventional Fourier transform described by Eq. (1). If the distance of propagation L is multiplied by $p \neq 1$, we obtain the fractional Fourier transform of order p expressed by the above Eq. (6).

2.2. Wigner distribution function

The Wigner distribution function is a function that describes the optical signals in the space coordinate and frequency domains, simultaneously. Based on the Wigner distribution, LOHMANN [5] reinvented the fractional Fourier transform. It appears that the description of signals in optical systems by means of Wigner distribution function resembles the ray concept in geometrical optics. If the signal is a two-dimensional image, the Wigner distribution function is then a four-dimensional function, because it is defined in the space and in the frequency domains, respectively. In the space domain, the Wigner distribution function is defined as

$$W(x,y;\xi,\eta) = \iint_{-\infty}^{\infty} u\left(x+\frac{x'}{2},y+\frac{y'}{2}\right)u^*\left(x-\frac{x'}{2},y-\frac{y'}{2}\right)\exp[-i2\pi(x'\xi+y'\eta)]\mathrm{d}x'\mathrm{d}y',$$

and in the frequency domain as

$$W(x,y;\xi,\eta) = \iint_{-\infty}^{\infty} \tilde{U}\left(\xi+\frac{\xi'}{2},\eta+\frac{\eta'}{2}\right)\tilde{U}^*\left(\xi-\frac{\xi'}{2},\eta-\frac{\eta'}{2}\right)\exp[i2\pi(x\xi'+y\eta')]\mathrm{d}\xi'\mathrm{d}\eta' \tag{7}$$

where the Fourier transform of the amplitude distribution $u(x,y)$ is denoted by $\tilde{U}(\xi,\eta) = \mathscr{F}\{u(x,y)\}$, and

$$|u(x,y)|^2 = \iint_{-\infty}^{\infty} W(x,y;\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta, \quad |\tilde{U}(\xi,\eta)|^2 = \iint_{-\infty}^{\infty} W(x,y;\xi,\eta)\mathrm{d}x\mathrm{d}y. \tag{8}$$

It is known that the Wigner distribution function is always real, and it is possible to express this function in the output plane in terms of the Wigner distribution function in the input plane. As we mentioned above, the simultaneous space-frequency description in the object or image space bears a close resemblance to the ray concept in geometrical optics, where the position (space coordinates (x,y)) and the direction (frequency (ξ,η)) of a ray are given simultaneously, too. If the coordinates in the Fourier plane of an optical setup are (x_F, y_F), then the frequencies of an investigated object will be expressed as

$$\xi = \frac{x_F}{\lambda f}, \quad \eta = \frac{y_F}{\lambda f} \tag{9}$$

where λ is the wavelength and f – the focal length of the lens.

The Wigner distribution function is very useful for description of any optical signals. If we have, for instance, a one-dimensional signal, then it is defined as

$$W(x;\xi) = \int_{-\infty}^{\infty} u\left(x+\frac{x'}{2}\right)u^*\left(x-\frac{x'}{2}\right)\exp(-i2\pi\xi x')\mathrm{d}x',$$

or

$$W(x;\xi) = \int_{-\infty}^{\infty} \tilde{U}\left(\xi+\frac{\xi'}{2}\right)\tilde{U}^*\left(\xi-\frac{\xi'}{2}\right)\exp(i2\pi\xi x')\mathrm{d}\xi'.$$

Here, we see that different transformations do not change the values of the Wigner distribution function. In fact, the free space propagation, passage through a lens and Fourier transformation have no influence on the Wigner distribution function, only the independent variables undergo a change. The free space propagation and transition through a lens correspond to a horizontal and a vertical shearing of the Wigner distribution function, respectively [5], but the conventional Fourier transform is expressed as $\Phi = 90°$ rotation of the Wigner distribution function in the (x, x_F) plane. If the first order Fourier transform is interpreted by the angle $\Phi = \pi/2$, then the p-th order fractional Fourier transform can be defined by the angle

$$\Phi = \frac{\pi}{2}p. \tag{10}$$

The coordinates of Wigner distribution function are connected with the angle Φ in relations

$$\begin{aligned} x' &= x\cos\Phi - x_F\sin\Phi, \quad & y' &= y\cos\Phi - y_F\sin\Phi, \\ x'_F &= x\sin\Phi + x_F\cos\Phi, \quad & y'_F &= y\sin\Phi + y_F\cos\Phi. \end{aligned} \tag{11}$$

Now, the fractional Fourier transform of a function $u(x, y)$ can be denoted as

$$\mathscr{F}^p\{u(x,y)\} = \tilde{U}_p(x_F, y_F),$$

and for the Wigner distribution function we have an expression

$$\begin{aligned} \mathscr{F}^p\{W(x,y;x_F,y_F)\} &= W_p(x,y;x_F,y_F) = W(x',y';x'_F,y'_F) \\ &= W(x\cos\Phi - x_F\sin\Phi, y\cos\Phi - y_F\sin\Phi; x\sin\Phi + x_F\cos\Phi, y\sin\Phi + y_F\cos\Phi). \end{aligned} \tag{12}$$

For example, conventional Fourier transform of the function takes place when $p = 1$, *i.e.*, $\Phi = \pi/2$; therefore we have

$$\mathscr{F}^1\{W(x,y;x_F,y_F)\} = W(-x_F, -y_F; x, y).$$

3. Fractional Fourier transform in lens system

The fractional Fourier transform as a generalization of the conventional Fourier transform can open up an important area of applications. It can be used not only for wave propagation and for signal processing, but also for applications in classical lens imaging systems.

We now consider an elementary optical system with the diffracting object placed in front of a lens and uniformly illuminated with normally incident monochromatic plane wave, as shown in Fig. 1. The source plane and its conjugate are located at infinity and in the back focal plane of the lens. The only restriction we impose is that the object must lie to the left of the lens and be illuminated from the left. To find the amplitude distribution of the field across the observation (output) plane of the system, the Fresnel diffraction formula is applied. If the field amplitude transmitted by the object is represented by the function $u_O(x_O, y_O)$, the output of the system may be written as

$$\tilde{U}(x_2, y_2) = \frac{1}{\lambda^2 z_1 z_2}\iint_{-\infty}^{\infty} \mathrm{d}x_O \mathrm{d}y_O \iint_{-\infty}^{\infty} u_O(x_O, y_O)\exp\left[i\frac{k}{2z_1}(x_O^2 + y_O^2)\right]$$

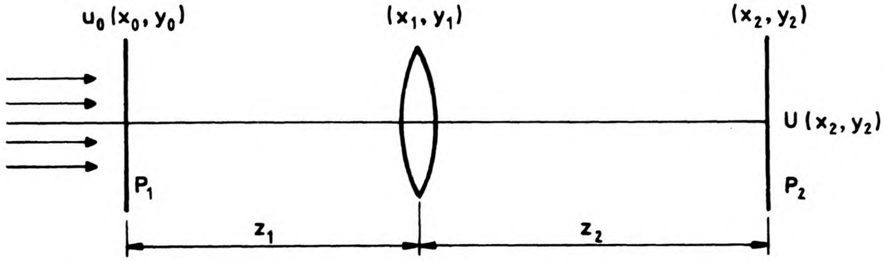

Fig. 1. Optical system with an object transparency in front of the lens for study of the Fourier transform operation. P_1, P_2 are the input and output planes, respectively

$$\times \exp\left[-i\frac{k}{z_1}(x_o x_1 + y_o y_1)\right] \exp\left[i\frac{k}{2}\left(\frac{1}{z_1}+\frac{1}{z_2}-\frac{1}{f}\right)(x_1^2+y_1^2)\right] \exp\left[i\frac{k}{2z_2}(x_2^2+y_2^2)\right]$$

$$\times \exp\left[-i\frac{k}{z_2}(x_1 x_2 + y_1 y_2)\right] \mathrm{d}x_1 \mathrm{d}y_1 \tag{13}$$

where $k = \frac{2\pi}{\lambda}$ is the wave number, and constant phase factor has been dropped since it does not affect the result in any significant way. Substituting the expression

$$\frac{1}{z_0} = \frac{1}{z_1} + \frac{1}{z_2} - \frac{1}{f}$$

into Eq. (13), we have

$$\tilde{U}(x_2, y_2) = \frac{\exp\left[i\frac{k}{2z_2}(x_2^2+y_2^2)\right]}{\lambda^2 z_1 z_2} \iint_{-\infty}^{\infty} \mathrm{d}x_o \mathrm{d}y_o \iint_{-\infty}^{\infty} u_o(x_o, y_o) \exp\left[i\frac{k}{2z_1}(x_o^2+y_o^2)\right]$$

$$\times \exp\left[i\frac{k}{2z_0}(x_1^2+y_1^2)\right] \exp\left\{-ik\left[x_1\left(\frac{x_o}{z_1}+\frac{x_2}{z_2}\right)+y_1\left(\frac{y_o}{z_1}+\frac{y_2}{z_2}\right)\right]\right\} \mathrm{d}x_1 \mathrm{d}y_1.$$

But

$$\iint_{-\infty}^{\infty} \exp\left[i\frac{k}{2z_0}(x_1^2+y_1^2)\right] \exp\left\{-i\frac{k}{z_1}\left[x_1\left(x_o+\frac{z_1}{z_2}x_2\right)+y_1\left(y_o+\frac{z_1}{z_2}y_2\right)\right]\right\} \mathrm{d}x_1 \mathrm{d}y_1$$

$$= C z_0 \exp\left\{-i\frac{kz_0}{2z_1^2}\left[\left(x_o+\frac{z_1}{z_2}x_2\right)^2+\left(y_o+\frac{z_1}{z_2}y_2\right)^2\right]\right\}.$$

If the complex valued constant C is neglected, and the quadratic terms in the exponent are expanded, then

$$\tilde{U}(x_2,y_2)=\frac{z_0}{\lambda^2 z_1 z_2}\exp\left[i\frac{kz_0(f-z_1)}{2z_1z_2f}(x_2^2+y_2^2)\right]\iint_{-\infty}^{\infty}u_O(x_O,y_O)$$
$$\times\exp\left[i\frac{kz_0(f-z_2)}{2z_1z_2f}(x_O^2+y_O^2)\right]\exp\left[-i\frac{kz_0}{z_1z_2}(x_Ox_2+y_Oy_2)\right]\mathrm{d}x_O\mathrm{d}y_O. \quad (14)$$

Figure 1 shows the basic configuration of the optical Fourier transform for any order. The amplitude distribution at the observation plane (x_2, y_2) may be found from a Fourier transform of the function

$$u_O(x_O,y_O)\exp\left[i\frac{kz_0(f-z_2)}{2z_1z_2f}(x_O^2+y_O^2)\right]$$

where the transform must be evaluated at frequencies

$$\xi=\frac{z_0x_2}{\lambda z_1z_2},\quad \eta=\frac{z_0y_2}{\lambda z_1z_2}.$$

When the exponent function of quadratic terms in the integral (14) is equal to unity for any value of z_1, *i.e.*, the condition $f-z_2=0$ is satisfied, then the first order Fourier transform of $u_O(x_O, y_O)$ occurs in the back focal length of the lens as an expression

$$\tilde{U}(x_2,y_2)=\frac{z_0}{\lambda^2 z_1 z_2}\exp\left[i\frac{k(f-z_1)}{2fz_2}(x_2^2+y_2^2)\right]\iint_{-\infty}^{\infty}u_O(x_O,y_O)$$
$$\times\exp\left[-i\frac{k}{z_2}(x_Ox_2+y_Oy_2)\right]\mathrm{d}x_O\mathrm{d}y_O. \quad (15)$$

At last we obtain an exact conventional Fourier transform relation by setting the second condition: $f-z_1=0$ in the above equation.

As mentioned, Lohmann's definition of the fractional Fourier transform is based on the Wigner distribution function of the signal and its Fourier transform. We compare the Wigner distributions of these functions by rotation of each other. Analogously to Eq. (15), the p-th order fractional Fourier transform of the function $u_O(x_O, y_O)$ is defined as [5]

$$\mathscr{F}^p\{u_O(x_O,y_O)\}=\tilde{U}_p(x_2,y_2)=C'\exp\left[i\frac{k(x_2^2+y_2^2)}{2f_1\tan\Phi}\right]\iint_{-\infty}^{\infty}u_O(x_O,y_O)\exp\left[i\frac{k(x_O^2+y_O^2)}{2f_1\tan\Phi}\right]$$
$$\times\exp\left[-i\frac{k}{f_1\sin\Phi}(x_Ox_2+y_Oy_2)\right]\mathrm{d}x_O\mathrm{d}y_O \quad (16)$$

where C' denotes an uninteresting constant factor that can be neglected, and the rotation angle Φ of the Wigner distribution function is connected with the fractional order p in Eq. (10). In other words, for a special case $p=1$, and the rotation of the underlying Wigner distribution function is equal to $\Phi = \pi/2$; we then obtain the conventional Fourier transform relation, as results from Eq. (16). The parameter $f_1 = f\sin\Phi$ is an arbitrary focal length and f is the focal length of the lens. Comparing the amplitude distribution at the output plane (as shown in Fig. 1) expressed by the two equations (14) and (16) we have

$$\frac{z_0(f-z_1)}{fz_1z_2} = \frac{\cos\Phi}{f\sin^2\Phi}, \qquad \frac{z_0(f-z_2)}{fz_1z_2} = \frac{\cos\Phi}{f\sin^2\Phi},$$

$$\frac{z_1z_2}{z_0} = f\sin^2\Phi, \qquad \frac{z_1z_2}{z_0} = f\sin^2\Phi,$$

$$\frac{f-z_1}{f} = \cos\Phi, \qquad \frac{f-z_2}{f} = \cos\Phi.$$

Hence $z_1 = z_2 = f(1-\cos\Phi)$, and the spatial frequencies at the fractional Fourier transform plane (x_2, y_2) are then expressed as

$$\xi = \frac{x_2}{\lambda z_2(1+\cos\Phi)}, \qquad \eta = \frac{y_2}{\lambda z_2(1+\cos\Phi)}. \tag{17}$$

In the Table we can find the values of the distance between the lens of the focal length $f = 100$ mm and the Fourier plane for different values of the fractional order in the range: $0 \leqslant p \leqslant 2$.

Table. Parameters of the lens for realizing the fractional Fourier transform (the distances in millimeters)

p	0	1/3	1/2	2/3	5/6	1	4/3	3/2	5/3	11/6	2
Φ	0	30°	45°	60°	75°	90°	120°	135°	150°	165°	180°
f_1	0	50.0	70.7	86.6	96.6	100.0	86.6	70.7	50.0	25.9	0
z_2	0	13.4	29.3	50.0	74.1	100.0	150.0	170.7	186.6	196.6	200.0

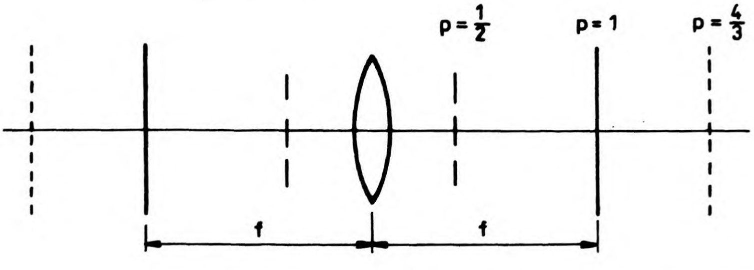

Fig. 2. Position of the three different fractional Fourier transform planes

In Figure 2, we see the Fourier planes of three different values of fractional order. The considerations show that a lens can produce the image of an object and/or a set of its Fourier transforms of different fractional orders. In a special case, when the condition: $\sin\Phi\tan\Phi = 1$ is satisfied, then Eq. (14) is similar to the Fresnel diffraction formula describing the amplitude distribution of optical field across the observation plane as a function of disturbance $u_O(x_O, y_O)$ in the input plane, by assuming free space between these planes. But the Fourier transform of fractional order $p = 0.576$ of the input is observed at the plane distance of $z_2 = 38.2$ mm behind the lens ($f = 100$ mm) and is expressed as

$$U(x_2, y_2) = C\exp\left[i\frac{k}{2f}(x_2^2+y_2^2)\right]\iint_{-\infty}^{\infty} u_O(x_O, y_O)\exp\left[i\frac{k}{2f}(x_O^2+y_O^2)\right]$$
$$\times\exp\left[-i\frac{kf}{z_2(2f-z_2)}(x_Ox_2+y_Oy_2)\right]\mathrm{d}x_O\mathrm{d}y_O.$$

The difference between the two operations lies in the scale of these transforms. The distance between the two planes in free space that is equal to f is described by Fresnel diffraction approximation and evaluated at frequencies $\xi = x_2/f\lambda$, $\eta = y_2/f\lambda$, whereas the fractional Fourier transform in the lens operation must be evaluated at frequencies

$$\xi = \frac{fx_2}{\lambda z_2(2f-z_2)}, \qquad \eta = \frac{fy_2}{\lambda z_2(2f-z_2)}.$$

The Table shows that the value of fractional order $p = 0$ corresponds to $z = 0$ and for this case the Fourier transform does not appear, but for $p = 2$ we obtain: $z_2 = 2f$ that describes the distance behind the lens where the bundle of rays emerging from an object point will cross an image point. It has been found that the scale of Fourier transform is a function of the axial position of the input object with nonparallel illumination. Let an object plane with the amplitude transmittance $u_0(x_O, y_O)$ be inserted in the front focal plane of the lens, and the illuminating point source at the point S on the optical axis at the distance z_S from the focal plane, as shown in Fig. 3. We see that the signal plane is now illuminated with a divergent bundle of light. The plane located at the image point P_I at a distance z_I from the back focal plane is the observation plane, where the Fourier transform of the signal appears. The point source located on the optical axis emits a divergent wave, which gives rise to disturbance at any point of the signal $u_O(x_O, y_O)$ in the front focal plane. A portion of the spherical wave front is collected by the object transparency and by the lens that transforms it in the form of a spherical wave converging towards the appropriate point in the observation plane at P_I. Using the paraxial approximation, the disturbance at any point in the front focal plane is given by $\exp\left[i\frac{k}{2z_S}(x_O^2+y_O^2)\right]$, where the constant phase factor has been omitted. Therefore the amplitude distribution just behind the object plane can be written as

$$u_S(x_O, y_O) = u_O(x_O, y_O)\exp\left[i\frac{k}{2z_S}(x_O^2+y_O^2)\right].$$

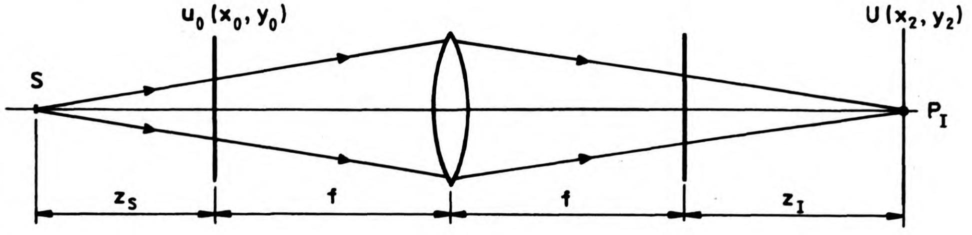

Fig. 3. Ray-tracing in the elementary optical system. S is an illuminating point source and P_I is its image; $u_O(x_O, y_O)$ is the amplitude transmittance in the front focal plane of the lens, $U(x_2, y_2)$ – the amplitude distribution of the field in the image plane of the imaging system

Using the Fresnel formula and the expression of the quadratic phase factor of the lens, the amplitude distribution behind the lens may be written as

$$u(x_1 y_1) = C(z) \iint_{-\infty}^{\infty} u_S(x_O, y_O) \exp\left[i\frac{k}{2f}(x_O^2+y_O^2\right]\exp\left[-i\frac{k}{f}(x_O, x_1)\right]dx_O dy_O,$$

and after applying the Fresnel diffraction integral in the image space, the amplitude distribution at the observation plane is expressed as

$$U(x_2 y_2) = C'(z)\exp\left[i\frac{k(x_2^2+y_2^2)}{2(f+z_I)}\right]\iint_{-\infty}^{\infty} u_S(x_O, y_O)\exp\left[i\frac{k}{2f}(x_O^2+y_O^2)\right]dx_O dy_O$$

$$\times\exp\iint_{-\infty}^{\infty}\exp\left[i\frac{k(x_1^2+y_1^2)}{2(f+z_I)}\right]$$

$$\times\exp\left\{-ik\left[x_1\left(\frac{x_O}{f}+\frac{x_2}{f+z_I}\right)+y_1\left(\frac{y_O}{f}+\frac{y_2}{f+z_I}\right)\right]\right\}dx_1 dy_1 \tag{18}$$

where the amplitude and phase of light at coordinates:

$$x_2' = \left(1+\frac{z_I}{f}\right)x_O+x_2, \qquad y_2' = \left(1+\frac{z_I}{f}\right)y_O+y_2$$

are related to the amplitude and phase of lens spectrum frequencies:

$$\xi = \frac{1}{\lambda}\left(\frac{x_O}{f}+\frac{x_2}{f+z_I}\right), \qquad \eta = \frac{1}{\lambda}\left(\frac{y_O}{f}+\frac{y_2}{f+z_I}\right).$$

The result of calculating the Fourier transform of the quadratic phase factor in the second integral of Eq. (18) leads to an expression of the amplitude distribution (18) proportional to the next operation of the Fourier transform in the following form:

$$\tilde{U}(x_2, y_2) = C'(z) \iint_{-\infty}^{\infty} u_O(x_O, y_O) \exp\left\{ i\frac{k}{2}\left[\left(\frac{1}{z_S} - \frac{z_I}{f^2}\right)(x_O^2 + y_O^2)\right]\right\}$$
$$\times \exp\left[-i\frac{k}{f}(x_O x_2 + y_O y_2)\right] \mathrm{d}x_O \mathrm{d}y_O. \tag{19}$$

The above equation shows that in the case when the phase curvature vanishes, amplitude distribution in the observation plane represents the fractional Fourier transform of the first order of the object transmittance (conventional), because of the condition

$$\frac{1}{z_S} - \frac{z_I}{f^2} = 0,$$

which for the fixed object and image planes expresses the Newton's equation: $z_S z_I = f^2$. The two conjugate planes coincide with the point source and its image observation planes, and simultaneously the Fourier transform operation between the transparency in the front focal plane and the observation plane can be performed. For a special case, if $z_S = f$, then $z_I = f$, too, and the distances of the two conjugate planes to the lens are equal. Now, we have the fractional Fourier transform of order $p = 2$, and the condition corresponds to formation of an inverted image that is rather inexact.

4. Conclusion

In this paper, an insight into optical implementation of fractional Fourier transform is provided. It has been discussed how to simplify the calculations of relations between the optical field amplitude distributions in different planes of an optical system. It has been shown that the optical implementation of the fractional Fourier transform based on Wigner distribution function is a more general operation than the conventional one, and can be applied also in optical systems of different configurations.

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