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Conventional and fractional optical Fourier transform

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In this paper, the Fourier transform of a spatially varying object for a lens and two lens systems is considered. As shown, the results of the Fourier transform depend on the relative positions of the input-output planes and the lens. Thus, the conventional Fourier transform can be extended to the fractional Fourier transform that corresponds to the amplitude distribution of the input signal whose Wigner distribution function shows a rotation of $\Phi = \frac{\pi}{2}p$, where p is the order of the fractional Fourier transform.

The Fourier transform of a spatially varying object determines the spatial frequency components of the object as long as the principal rays are close enough to the optical axis, so that nonlinear terms of the field angles may be neglected. The restriction to small field angles imposes severe limitations of useful range of Fourier lens. If such a lens transforms the amplitude distribution of an object from the front to the back focal planes, the field distribution on the screen located at the back focal plane becomes exactly the Fourier transform of the input function. Amplitude distribution described by Fourier transform multiplied by a quadratic phase term: $\exp\left[i\frac{k}{2z_2}(x^2+y^2)\right]$ is referred to as imperfect Fourier transform. But the conventional Fourier transform can be extended to the fractional (or generalized) Fourier transform [1]–[3] that corresponds to the amplitude distribution of the input signal whose Wigner distribution shows a rotation of $\Phi = \frac{\pi}{2}p$.

The converging beam Fourier transform is based on the diffraction of a converging spherical wave by object spatial frequency components, as shown in Fig. 1. In this case, the object is behind the lens which is usually corrected only for spherical aberration to produce a perfect spherical wave. If the lens is illuminated by a normally incident plane wave, then the spherical wave is incident on the object transparency and after diffraction forms a field distribution across the back focal plane of the lens described by an expression proportional to the Fourier transform of its transmittance

$$\mathscr{F}\{U_o(x_o y_o)\} = \frac{A\exp\left[i\frac{k}{2z_2}(x^2+y^2)\right]}{i\lambda z_2}\iint_{-\infty}^{\infty} U_o(x_o,y_o)\exp\left[-i\frac{k}{z_2}(x_o x+y_o y)\right]\mathrm{d}x_o\mathrm{d}y_o.$$

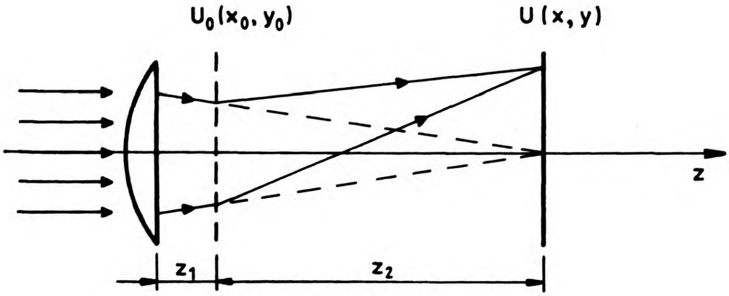

Fig. 1. Fourier transform realization of the input transparency placed behind the lens

Thus, up to quadratic phase factor, the field distribution in the back focal plane is the Fourier transform of that portion of the object subtended by the projected lens aperture. In order to eliminate the phase factor associated with the Fourier transform, a converging (holographic) lens of focal length $f = z_2$ is placed at the output plane. The results of the Fourier transform depend on the relative positions of the input plane, the lens and the output plane. The setup allows the size of spectrum to be varied in this way that by increasing the distance z_2, the spatial size of the transform is increased, and by decreasing the distance, it is made smaller. This flexibility can be of considerable utility in Fourier transform realization, especially in spatial filtering applications.

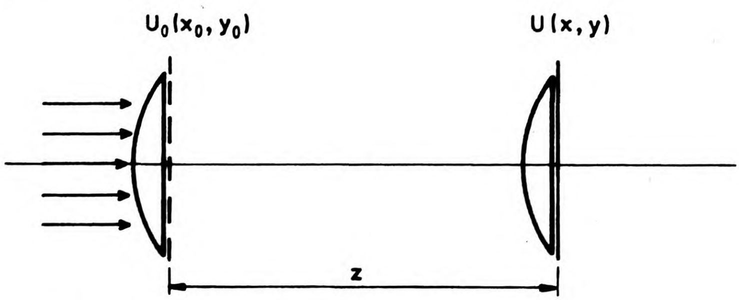

Fig. 2. Fourier transform in the double lens system

For an optical system composed of two lenses of the focal length f_1 and f_2, respectively, with the interlens distance being z, the object transparency is placed just behind the first lens and the output (observation) plane behind the second lens, as shown in Fig. 2. When the coherent plane wave enters the system in parallel to its optical axis (see Fig. 2), the distribution of light at the output plane is

$$U(x,y) = \frac{A\exp\left[i\frac{k}{2}\left(\frac{1}{z}-\frac{1}{f_2}\right)(x^2+y^2)\right]}{i\lambda z}\iint_{-\infty}^{\infty} U_O(x_O,y_O)\exp\left[i\frac{k}{2}\left(\frac{1}{z}-\frac{1}{f_1}\right)(x_O^2+y_O^2)\right]\times$$

$$\times \exp\left[-i\frac{k}{z}(x_O x + y_O y)\right] dx_O dy_O.$$

In the special case, when $z = f_1 = f_2 = f$, the plane of the first lens covers the front focal plane of the second lens, while the plane of the second lens covers the back focal plane of the first lens. The amplitude distribution at the output of the system represents then the Fourier transform of the amplitude distribution at the input, which is to say the first order ($p = 1$) of fractional Fourier transform. If $z \neq f_1 = f_2$, then te amplitude distribution at the observation plane is expressed as

$$U(x,y) = \frac{1}{iz\lambda}\iint_{-\infty}^{\infty} U_O(x_0, y_O) \exp\left[i\frac{k(f-z)}{2zf}[(x_O^2 + x^2) + (y_O^2 + y^2)]\right]$$
$$\times \exp\left[-i\frac{k}{z}(x_O x + y_O y)\right] dx_O dy_O,$$

and the fractional Fourier transform can be implemented. Thus, an optical system of two lenses each having the focal length f is separated by $z = (1 - \cos\Phi)$, as shown by LOHMANN [4]. Such a system of two lenses is more compact for Fourier transform operation than the one lens system and is more profitable for applications.

References

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