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Exact *N*-envelope-soliton solutions of the Hirota equation

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We discuss some properties of the soliton equations of the type $\partial u/\partial t = S[u, \bar{u}]$, where S is a nonlinear operator differential in x, and present the additivity theorems of the class of the soliton equations. On using the theorems, we can construct a new soliton equation through two soliton equations with similar properties. Meanwhile, exact *N*-envelope-soliton solutions of the Hirota equation are derived through the trace method.

Keywords: exact solutions, Hirota equation, solitons.

The trace method, which has been applied to the Korteweg–de Vries equation [1], modified Korteweg–de Vries equation [2], Kadomtsev–Petviashvili equation [3], sine-Gordon equation [4], [5] and Gz Tu equation [6], is useful for understanding these equations. The *N*-soliton solutions and some other results of these equations [7] have been derived through the trace method.

The present paper deals with an application of the trace method to the nonlinear partial differential equation as follows:

$$\frac{\partial}{\partial t}u + L_x u = N_x(u, \bar{u}) \tag{1}$$

where:

$$L_x u = \sum_{k=0}^{N_1} \alpha_k \frac{\partial^k}{\partial x^k} u,$$

$$N_x(u, \bar{u}) = \sum_{k=1}^{N_2} \beta_k \prod_{m=0}^{N_k} \left(\frac{\partial^m}{\partial x^m} u\right)^{r_{m,k}} \left(\frac{\partial^m}{\partial x^m} \bar{u}\right)^{s_{m,k}}$$

where α_k, β_k are complex constants, $r_{m,k}$, $s_{m,k}$ are nonnegative integers, $r_k = \sum_{m=0}^{N_k} r_{m,k}$; $s_k = \sum_{m=0}^{N_k} s_{m,k}$; $r_1 = r_2 = \ldots = r_{N_2} = r$; $s_1 = s_2 = \ldots = s_{N_2} = s$ and $d = r + s \geq 2$; r, s satisfy one of the relations:

$$s \geq 1 \quad \text{for} \quad r = s + 1, \tag{i}$$

$$s = 0 \quad \text{for} \quad r \geq 2. \tag{ii}$$

Substituting the formal series

$$u = u^{(1)} + u^{(d)} + \ldots + u^{((d-1)n+1)} + \ldots \tag{2}$$

into Eq. (1), we obtain a set of equations for $u^{((d-1)n+1)}$ $(n = 0, 1, 2, \ldots)$:

$$\left(\frac{\partial}{\partial t} + L_x\right) u^{((d-1)n+1)} = \sum_{l_1=1}^{N} \cdots \sum_{l_{(d-1)n+1}=1}^{N} C^{(n)}(P_{l_1}, \bar{P}_{l_2}, \ldots, \bar{P}_{l_{(d-1)n}}, P_{l_{(d-1)n+1}}) \times \phi_{l_1}^2 \bar{\phi}_{l_2}^2 \cdots \bar{\phi}_{l_{(d-1)n}}^2 \phi_{l_{(d-1)n+1}}^2, \tag{3}$$

$$\left(\frac{\partial}{\partial t} + L_x\right) u^{((r-1)n+1)} = \sum_{l_1=1}^{N} \cdots \sum_{l_{(r-1)n+1}=1}^{N} C^{(n)}(P_{l_1}, P_{l_2}, \ldots, P_{l_{(r-1)n}}, P_{l_{(r-1)n+1}}) \times \phi_{l_1}^2 \phi_{l_2}^2 \cdots \phi_{l_{(r-1)n}}^2 \phi_{l_{(r-1)n+1}}^2 \tag{4}$$

where Eqs. (3) and (4) correspond to relations (i) and (ii), respectively,

$$\phi_k(x, t) = A_k(0)\exp(P_k x - \Omega_k t), \quad \Omega_k = \frac{1}{2} L_p(2P_k),$$

$$L_p(x) = \sum_{k=0}^{N_1} \alpha_k x^k,$$

$A_k(0)$ and P_k are complex constants $(k = 1, 2, \ldots, N)$, and $C^{(0)} = 0$.

We can obtain the solutions for Eqs. (3) and (4) in the following form:

$$u^{((d-1)n+1)} = \sum_{l_1=1}^{N} \cdots \sum_{l_{(d-1)n+1}=1}^{N} \pi^{(n)} \phi_{l_1}^2 \bar{\phi}_{l_2}^2 \cdots \bar{\phi}_{l_{(d-1)n}}^2 \phi_{l_{(d-1)n+1}}^2, \tag{5}$$

$$u^{((r-1)n+1)} = \sum_{l_1=1}^{N} \cdots \sum_{l_{(r-1)n+1}=1}^{N} \pi^{(n)} \phi_{l_1}^2 \phi_{l_2}^2 \cdots \phi_{l_{(r-1)n}}^2 \phi_{l_{(r-1)n+1}}^2 \tag{6}$$

where $\pi^{(0)} = 1$ and

$$\begin{aligned}\pi^{(n)} &= C^{(n)}/[L_p(2P_{l_1} + 2\bar{P}_{l_2} + \ldots + 2\bar{P}_{l_{(d-1)n}} + 2P_{l_{(d-1)n+1}}) \\ &\quad - L_p(2P_{l_1}) - \bar{L}_p(2\bar{P}_{l_2}) - \ldots - \bar{L}_p(2\bar{P}_{(d-1)n}) - L_p(2P_{l_{(d-1)n+1}})]\end{aligned} \tag{7}$$

or

$$\begin{aligned}\pi^{(n)} &= C^{(n)}/[L_p(2P_{l_1} + 2P_{l_2} + \ldots + 2P_{l_{(r-1)n}} + 2P_{l_{(r-1)n+1}}) \\ &\quad - L_p(2P_{l_1}) - L_p(2P_{l_2}) - \ldots - L_p(2P_{l_{(r-1)n}}) - L_p(2P_{l_{(r-1)n+1}})].\end{aligned} \tag{8}$$

Theorem 1. Let

$$\frac{\partial u}{\partial t} + L_x' u = N_x'(u, \bar{u}), \qquad \frac{\partial u}{\partial t} + L_x'' u = N_x''(u, \bar{u})$$

be two arbitrary equations that are defined by Eq. (1). If $r' = r''$, $s' = s''$, $\pi'^{(n)} = \pi''^{(n)}$ $(n = 0, 1, 2, \ldots)$, then, for equation

$$\frac{\partial u}{\partial t} + L_x^* u = N_x^*(u, \bar{u})$$

(where $L_x^* = aL_x' + bL_x''$, $N_x^*(u, \bar{u}) = aN_x'(u, \bar{u}) + bN_x''(u, \bar{u})$, and a, b are two arbitrary real numbers), we have $\pi^{*(n)} = \pi'^{(n)} = \pi''^{(n)}$ $(n = 0, 1, 2, \ldots)$.

Proof. We consider the case (ii) by mathematical induction. Obviously $\pi^{*(0)} = \pi'^{(0)} = \pi''^{(0)} = 1$. Assume $\pi^{*(n)} = \pi'^{(n)} = \pi''^{(n)}$ $(n = 0, 1, 2, \ldots, k)$. When $n = k + 1$, from Eq. (4), $C^{*(k+1)} = aC'^{(k+1)} + bC''^{(k+1)}$, and from Eq. (8)

$$\begin{aligned}\pi^{*(k+1)} &= C^{*(k+1)}\Big/\left[L_p^*\left(2\sum_{m=1}^{(r-1)n+1} P_{l_m}\right) - \sum_{m=1}^{(r-1)n+1} L_p^*(2P_{l_m})\right] \\ &= [aC'^{(k+1)} + bC''^{(k+1)}]\Big/\left[aL_p'\left(2\sum_{m=1}^{(r-1)n+1} P_{l_m}\right) + bL_p''\left(2\sum_{m=1}^{(r-1)n+1} P_{l_m}\right)\right. \\ &\quad \left. - a\sum_{m=1}^{(r-1)n+1} L_p'(2P_{l_m}) - b\sum_{m=1}^{(r-1)n+1} L_p''(2P_{l_m})\right] = \pi'^{(k+1)} = \pi''^{(k+1)}.\end{aligned}$$

For the case (i), we can prove it in the same manner.

We introduce two $N \times N$ matrices B and D whose elements are given respectively by $B_{mn} = [1/(P_m + P_n)]\phi_m(x, t)\phi_n(x, t)$, $D_{mn} = [1/(P_m + \bar{P}_n)]\phi_m(x, t)\bar{\phi}_n(x, t)$.

Theorem 2. Let

$$\frac{\partial u}{\partial t} + L'_x u = N'_x(u, \bar{u}), \quad \frac{\partial u}{\partial t} + L''_x u = N''_x(u, \bar{u})$$

be two arbitrary equations that are defined by Eq. (1). If they have respective solutions

$$u' = \mathrm{Tr}[B'_x f(D'\bar{D}')] \ (\text{or } \mathrm{Tr}[B'_x g(B')]),$$

$$u'' = \mathrm{Tr}[B''_x f(D''\bar{D}'')] \ (\text{or } \mathrm{Tr}[B''_x g(B'')])$$

where f, g are arbitrarily derivable functions in the neighbourhood of zero, then, for equation

$$\frac{\partial u}{\partial t} + L^*_x u = N^*_x(u, \bar{u})$$

(where $L^*_x = aL'_x + bL''_x$, $N^*_x(u, \bar{u}) = aN'_x(u, \bar{u}) + bN''_x(u, \bar{u})$, and a, b are two arbitrary real numbers), we have solution

$$u^* = \mathrm{Tr}[B^*_x f(D^*\bar{D}^*)] \ (\text{or } \mathrm{Tr}[B^*_x g(B^*)]).$$

Proof. Since f, g are arbitrarily derivable functions in the neighbourhood of zero, f, g can be expanded into power series in convergence region. Correspondingly, u', u'' can be expanded into power series. Comparing the coefficients, we have $r' = r''$, $s' = s''$, $\pi'^{(n)} = \pi''^{(n)}$ ($n = 0, 1, 2, \ldots$). From Theorem 1, we obtain

$$u^* = \mathrm{Tr}[B^*_x f(D^*\bar{D}^*)] \ (\text{or } \mathrm{Tr}[B^*_x g(B^*)]).$$

On using Theorems 1 and 2, we can construct a new soliton equation through two soliton equations with similar properties. As an example, we use the trace method to solve the Hirota equation [8] as follows:

$$i\psi_t + i3\alpha|\psi|^2\psi_x + \rho\psi_{xx} + i\sigma\psi_{xxx} + \delta|\psi|^2\psi = 0 \tag{9}$$

where α, ρ, σ and δ are positive real constants with the relation $\alpha/\sigma = \delta/\rho = \lambda$. In one limit of $\alpha = \sigma = 0$, the equation becomes the nonlinear Schrödinger equation [9] that describes a plane self-focusing and one-dimensional self-modulation of waves in nonlinear dispersive media

$$i\psi_t + \rho\psi_{xx} + \delta|\psi|^2\psi = 0. \tag{10}$$

In another limit of $\rho = \delta = 0$ the equation for real ψ, becomes the modified Korteweg–de Vries equation [10], [11]

$$\psi_t + 3\alpha\psi^2\psi_x + \sigma\psi_{xxx} = 0. \tag{11}$$

Hence, the present solutions reveal the close relation between classical solitons and envelope solitons. Substituting the formal series

$$\psi = \psi^{(1)} + \psi^{(3)} + \ldots + \psi^{(2n+1)} + \ldots \tag{12}$$

into Eq. (9), we obtain a set of equations for $\psi^{(2k+1)}$ $(k = 0, 1, 2, \ldots)$:

$$i\psi_t^{(1)} + \rho\psi_{xx}^{(1)} + i\sigma\psi_{xxx}^{(1)} = 0, \tag{13}$$

$$i\psi_t^{(3)} + \rho\psi_{xx}^{(3)} + i\sigma\psi_{xxx}^{(3)} = -i3\alpha\psi^{(1)}\,\bar{\psi}^{(1)}\,\psi_x^{(1)} - \delta\,\psi^{(1)}\,\bar{\psi}^{(1)}\,\psi^{(1)}, \tag{14}$$

$$\vdots$$

$$i\psi_t^{(2n+1)} + \rho\psi_{xx}^{(2n+1)} + i\sigma\psi_{xxx}^{(2n+1)}$$

$$= -i3\alpha\sum_{l=0}^{n-1}\sum_{m=0}^{n-l-1}\psi^{(2l+1)}\bar{\psi}^{(2m+1)}\psi_x^{(2n-2l-2m-1)}$$

$$-\delta\sum_{l=0}^{n-1}\sum_{m=0}^{n-l-1}\psi^{(2l+1)}\bar{\psi}^{(2m+1)}\psi^{(2n-2l-2m-1)} \tag{15}$$

$$\vdots$$

We can solve the set of equations iteratively:

$$\psi^{(1)} = \sum_{l_1=1}^{N}\phi_{l_1}^2(x, t), \tag{16}$$

$$\psi^{(3)} = -\frac{\lambda}{8}\sum_{l_1=1}^{N}\sum_{l_2=1}^{N}\sum_{l_3=1}^{N}\frac{1}{(P_{l_1}+\bar{P}_{l_2})(\bar{P}_{l_2}+P_{l_3})}\phi_{l_1}^2(x, t)\bar{\phi}_{l_2}^2(x, t)\phi_{l_3}^2(x, t) \tag{17}$$

where $\phi_k(x, t) = A_k(0)\exp(P_k x - \Omega_k t)$, $\Omega_k = -2i\rho P_k^2 + 4\sigma P_k^3$, $A_k(0)$ and P_k are complex constants relating respectively to the amplitude and phase of the k-th soliton

($k = 1, 2, ..., N$). We introduce two $N \times N$ matrices B and D whose elements are given respectively by:

$$B_{mn} = \left[\frac{1}{P_m + P_n}\right] \phi_m(x, t)\phi_n(x, t), \qquad D_{mn} = \left[\frac{1}{P_m + \bar{P}_n}\right] \phi_m(x, t)\bar{\phi}_n(x, t).$$

With matrices B and D, $\psi^{(1)}$ and $\psi^{(3)}$ are expressed as:

$$\psi^{(1)} = \mathrm{Tr}\,[B_x], \tag{18}$$

$$\psi^{(3)} = -\frac{\lambda}{8}\mathrm{Tr}\,[B_x(D\bar{D})]. \tag{19}$$

In general, we can prove that

$$\psi^{(2n+1)} = (-1)^n \frac{\lambda^n}{8^n}\,\mathrm{Tr}\,[B_x(D\bar{D})^n], \qquad n = 0, 1, 2, ... \tag{20}$$

satisfies Eq. (15).

With the definitions of matrices B and D

$$\psi^{(2n+1)} =$$

$$= (-1)^n \frac{\lambda^n}{8^n} \sum_1 \cdots \sum_{2n+1} \frac{\phi_1^2\bar{\phi}_2^2\cdots\bar{\phi}_{2n}^2\phi_{2n+1}^2}{(P_1 + \bar{P}_2)(\bar{P}_2 + P_3)\dots(P_{2n-1} + \bar{P}_{2n})(\bar{P}_{2n} + P_{2n+1})}. \tag{21}$$

Here and in the following we simplify the expressions by writing $1, 2, ..., 2n + 1$ instead of $l_1, l_2, ..., l_{2n+1}$. There should be no confusion about this. We have

$$i\psi_t^{(2n+1)} + \rho\psi_{xx}^{(2n+1)} + i\sigma\psi_{xxx}^{(2n+1)}$$

$$= (-1)^n \frac{\lambda^n}{2^{3n-1}} \sum_1 \cdots \sum_{2n+1} \{4i\sigma[(P_1 + \bar{P}_2 + \dots + \bar{P}_{2n} + P_{2n+1})^3$$

$$- (P_1^3 + \bar{P}_2^3 + \dots + \bar{P}_{2n}^3 + P_{2n+1}^3)] + 2\rho[(P_1 + \bar{P}_2 + \dots + \bar{P}_{2n} + P_{2n+1})^2$$

$$- (P_1^2 - \bar{P}_2^2 + \dots - \bar{P}_{2n}^2 + P_{2n+1}^2)]\}$$

$$\times \frac{\phi_1^2\bar{\phi}_2^2\cdots\bar{\phi}_{2n}^2\phi_{2n+1}^2}{(P_1 + \bar{P}_2)(\bar{P}_2 + P_3)\dots(P_{2n-1} + \bar{P}_{2n})(\bar{P}_{2n} + P_{2n+1})}. \tag{22}$$

Substituting two identities

$$(k_1 + k_2 + \ldots + k_{2n} + k_{2n+1})^3 - (k_1^3 + k_2^3 + \ldots + k_{2n}^3 + k_{2n+1}^3)$$

$$= 3\sum_{l=0}^{n-1} \sum_{m=0}^{n-l-1} [(k_1 + \ldots + k_{2l+1})(k_{2l+1} + k_{2l+2})(k_{2l+2m+2} + k_{2l+2m+3})$$

$$+ (k_{2l+1} + k_{2l+2})(k_{2l+2m+2} + k_{2l+2m+3})(k_{2l+2m+3} + \ldots + k_{2n+1})], \quad (23)$$

$$(k_1 + k_2 + \ldots + k_{2n} + k_{2n+1})^2 - (k_1^2 - k_2^2 + \ldots - k_{2n}^2 + k_{2n+1}^2)$$

$$= 2\sum_{l=0}^{n-1} \sum_{m=0}^{n-l-1} [(k_{2l+1} + k_{2l+2})(k_{2l+2m+2} + k_{2l+2m+3})] \quad (24)$$

into Eq. (22) and using Eq. (20) for $\psi^{(2k+1)}$ $(k < n)$, we obtain

$$i\psi_t^{(2n+1)} + \rho\psi_{xx}^{(2n+1)} + i\sigma\psi_{xxx}^{(2n+1)}$$

$$= -\delta\sum_{l=0}^{n-1} \sum_{m=0}^{n-l-1} \psi^{(2l+1)}\bar{\psi}^{(2m+1)}\psi^{(2n-2l-2m-1)}$$

$$- i\frac{3}{2}\alpha\sum_{l=0}^{n-1} \sum_{m=0}^{n-l-1} \bar{\psi}^{(2m+1)}(\psi^{(2l+1)}\psi^{(2n-2l-2m-1)})_x.$$

Therefore we obtain the N-envelope-soliton solution for Eq. (9) in the following form:

$$\psi = \mathrm{Tr}\left\{\sum_{k=0}^{\infty} (-1)^k \frac{\lambda^k}{8^k} [B_x(D\bar{D})^k]\right\} = \mathrm{Tr}\left[B_x\left(1 + \frac{\lambda}{8} D\bar{D}\right)^{-1}\right] \quad (25)$$

where $\|D\bar{D}\| < 8/\lambda$ in a certain region. In particular, for $N = 1$, we obtain the one-envelope-soliton solution

$$\psi(x, t) = \frac{A_1(0)}{2} \operatorname{sech}[(P_1 + \bar{P}_1)x - (\Omega_1 + \bar{\Omega}_1)t + \eta] \times \exp[(P_1 - \bar{P}_1)x - (\Omega_1 - \bar{\Omega}_1)t - \eta] \tag{26}$$

where

$$\eta = \frac{1}{2} \ln\left(\frac{\lambda|A_1(0)|^4}{8(P_1 + \bar{P}_1)^2}\right).$$

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