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Conditions sufficient for a one-dimensional unique recovery of the phase under assumption that the image intensity distributions: $|f(x)|^2$ and $\left|\frac{df(x)}{dx}\right|^2$ are known

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It has been shown that the knowledge of intensity distributions $|f(x)|^2$ and $\left|\frac{df(x)}{dx}\right|^2$ in the image plane, where $f(x)$ is the complex amplitude distribution, suffices to determine uniquely the respective phase distribution. The recovery is then reduced to determining the cut-off frequency and stating whether the intensity coming from the spatial frequency spectrum at a point coordinate equal to the right-hand cut-off frequency is greater (or less) than the intensity at the point of coordinates equal to the left-hand cut-off frequency.

Introduction

As it is well known the coherent field is described by two magnitudes: amplitude and phase comprised in the complex amplitude of the form

$$f(x) = A(x)\exp[i\varphi(x)]. \tag{1}$$

If the measurement in the image plane involves the square-law detector, only the intensity distribution (proportional to the squared modulus of the complex amplitude) is recorded and thus the phase information gets usually lost. This means that if no additional information about the form of $f(x)$ is available, the knowledge of the amplitude $A(x)$ does not allow to reconstruct the phase distribution $\varphi(x)$. A radical improvement of the situation occurs if it is known a priori that $f(x)$ is a band-limited signal. This assumption enables the recovery of all distributions of complex amplitude, provided that they are band-limited and have amplitudes equal to $A(x)$ [1]. Unfortunately, there exist an infinite number (continuum) of such distributions. The degree of nonuniqueness of phase recovery depends upon the zero-places distributions on the complex plane of the analytic extension $f(z)$ of the complex amplitude distribution $f(x)$.

If the number of zero places in the upper or lower complex half-plane is finite the number of admissible solutions is at most countable. The unique determination of phase with the accuracy to a linear component $a+bx$ is possible when the upper or lower half-plane is free from zero-places [2]. This fact cannot be stated if only $A^2(x)$ is known. An exception presents the case when $f(x)$ has only real zero-places. However, the class of complex amplitude distributions, which have only real zero-places is narrow [3]. So far, the physically realizable criteria stating wheather $f(z)$

has only real zero places are not known either. Although it is possible to determine $f(z)\cdot f^*(z^*)$ from $A^2(x)$ and next to find whether $f(z)$ has real zero-places, such a procedure, however, would require infinite number of steps being, therefore, of no practical importance. A part from the knowledge of amplitude distribution $A(x)$ and the fact that $f(x)$ is band-limited either additional a priori information or an additional measurement is only necessary, the latter should restrict the class of admissible complex amplitude distributions that the phase recovery within this class be unique. In this way, the knowledge of $|f(x)|^2$ and $|F(\omega)|^2$, where $F(\omega)$ is the spatial frequency spectrum of $f(x)$, suffices to perform a unique phase recovery, if $F(\omega)$ is an analytic function [4]. The unique recovery of the phase is also possible, when the intensity distribution is known in the image plane of a microscopic system before and after defocussing this system [5].

The method proposed below assures also a unique phase recovery. Similarly, as it is the case in the methods [4, 5] mentioned above, the uniqueness is obtained at the expense of an additional measurement. In this case besides the intensity $A^2(x)$ also $\left|\frac{df(x)}{dx}\right|^2$ is measured.

Sufficient conditions for unique phase recovery

Let us assume that we are able to realize the operation $\left|\frac{df(x)}{dx}\right|$. In the face of (1)

$$f'(x) = \big(A'(x)+i\varphi'(x)A(x)\big)\exp[i\varphi(x)], \tag{2}$$

where prime denotes the differentiation with respect to x variable.

Assume further that the intensity distributions corresponding to $|f(x)|^2$ and $|f'(x)|^2$ are known. Then from eqs. (1) and (2) it is possible to determine the squared modulus of the phase derivative as related to the measurable quantities $|f(x)|^2$ and $|f'(x)|^2$:

$$|\varphi'(x)|^2 = \frac{|f'(x)|^2-\big(|f(x)|'\big)^2}{|f(x)|^2}. \tag{3}$$

However, from (3) the phase cannot be determined uniquely. The determination of $|f'(x)|$ only does not allow to conclude whether any change of sign at the zero-places of $\varphi'(x)$ has occured. Therefore, if there exist n zero-places of $|\varphi'(x)|$ the function $\varphi(x)$ may be recovered in 2^{n+1} variants*. Even if $\varphi'(x)$ has no zero-places the two following solutions

$$\varphi_1(x) = \int |\varphi'(x)|\,dx, \tag{4a}$$

$$\varphi_2(x) = -\int |\varphi'(x)|\,dx \tag{4b}$$

are still undistinguishable.

Thus, in the general case, the knowledge of both $|f(x)|^2$ and $|f'(x)|^2$ appears to be insufficient to a unique determination of the phase.

* If no additional information concerning the $\varphi'(x)$ distribution is available.

The situation is radically improved, if the assumptions of the following theorem are fulfilled.

Theorem

If $f(x)$ is i) a band-limited function of known cut-off frequency ω_0, and ii) the number of zero-places of the analytic extension of the functions $f(x)$ and $f'(x)$ laying in the upper complex half-planes is finite then the system of equations

$$\begin{aligned} |g(x)| &= |f(x)|, \\ |g'(x)| &= |f'(x)| \end{aligned} \tag{5}$$

has only one solution (with the accuracy to an $\exp[i \cdot a]$ factor), which has also the above properties i), ii). This solution is of the form

$$g(x) = f(x) \exp[i \cdot a]. \tag{6}$$

Proof:

Each band-limited function, which has a finite number of zero-places in the upper half-plane, and the amplitude equal to $|f(x)|$ is of the form

$$\dot{f}_{(n_k)}(x) = f(x) e^{i(a+bx)} \prod_{n=n_k} \frac{x-z_n^*}{x-z_n}, \tag{7}$$

where $\{z_{n_k}\}$ is an arbitrary finite subset of the set of the zero-places of the function of the complex variable

$$f(z) = \int_{-\omega_0}^{\omega_0} F(\omega) e^{2\pi i \omega z} \, d\omega, \tag{8}$$

where $F(\omega)$ is the spatial frequency spectrum of the complex amplitude $f(x)$ [1]. The asterisk in the formula (7) denotes the complex conjugate. Similarly, each band-limited function, which has a finite number of zeros in the upper half-plane and has the amplitude equal to $|f'(x)|$ is of the form

$$d_{(m_l)}(x) = f'(x) e^{i(c+dx)} \prod_{m=m_l} \frac{x-w_m^*}{x-w_m}. \tag{9}$$

where $\{w_{m_l}\}$ is an arbitrary finite subset of the set of zero-places of the function

$$f'(z) = \int_{-\omega_0}^{\omega_0} 2\pi i \omega F(\omega) e^{2\pi i \omega z} d\omega. \tag{10}$$

It is well known that when multiplying any function by the Blaschke factor we do not change the cut-off frequency ω_0. Only the non-zero value of b in (7) would cause a change of the cut-off frequency from $\pm\omega_0$ to $\pm\omega_0 + \frac{b}{2\pi}$. Therefore, we assume $b = 0$ to enable the function $f_{(n_k)}(x)$ to fulfil the assumption that the cut-off frequency ω_0 is known. For the same reasons we put $d = 0$ in (9).

Assume that there exists such a function $g(x)$ which fulfills simultaneously the asumptions of our theorem and eqs. (5). Then there exist also finite subsets

$\{z_{n_k}\}$ and $\{w_{m_l}\}$ such that

$$g(x) = f_{(n_k)}(x), \tag{10a}$$

$$g'(x) = d_{(m_l)}(x). \tag{10b}$$

Substituting (10a) into (10b) and next taking advantage of (7) and (9) we get the following differential equation

$$f'(x)\left[e^{ia}\prod_{n=n_k}\frac{x-z_n^*}{x-z_n} - e^{ic}\prod_{m=m_k}\frac{x-w_m^*}{x-w_m}\right] + f(x)e^{ia}\left(\prod_{n=n_k}\frac{x-z_n^*}{x-z_n}\right)' = 0. \tag{11}$$

This equation will be fulfilled also across the whole complex plane, owing to the fact that each of the functions in eqs. (10a, b) may be uniquely extended on the whole complex plane preserving the equality sign. As the products in (11) are finite equation (11) may be put in the form*

$$\frac{f(z)}{f(z)} = Q(z), \tag{12}$$

where $Q(z)'$ is the rational function, i.e. the quotient of the polynomials. By integrating both sides of eq. (12) across an arbitrary circle K surrounding all the poles of the function $Q(z)$ we obtain

$$\frac{1}{2\pi i}\int_K \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_K Q(z)\,dz. \tag{13}$$

From the residuum theorem (c.f. [6] section 3.11) it follows that the right-hand side of (13) is equal to the finite sum of residua and thus it will be constant with the increment of the radius of K circle. The left-hand side will be equal to the number of zeros of the function $f(z)$ lying within the circle K due to the fact that the band-limited function has no poles (cf. [6], section 3.4). Since the function $f(z)$, being a band-limited function, has an infinite number of zeros, the left-hand side of the eq. (13) will tend to infinity with the respective increase of the circle radius. In this way as a consequence of the assumption (10a, b) we get an inconsistance which completes the proof of the above theorem. This inconsistance will disappear if the Blaschke factors disappear in (7), and consequently in (9). Then, putting $c = a$ we obtain (6).

It is evident that the theorem will remain true when the functions $f(z)$ and $f'(z)$ have the finite numbers of zero-places in the lower complex half-plane.

A possibility of practical realization

In order to take practical advantage of the above results it is necessary to have a possibility of optical performing of the derivative $f'(z)$. It may be realized in such optical systems, in which the frequency plane is available for manipulation, i.e. such that

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* We assume that $f(z) \not\equiv 0$ because if it is not case the eqs. (5) have, of course, a unique solution.

a transparency could be placed in it. Then it will suffice to locate there a transparency of amplitude transmittance proportional to

$$T(\omega) = 2\pi i\omega \cdot \mathrm{rect}\left(\frac{\omega}{2\omega_0}\right). \tag{14}$$

Such a transparency may be, for instance, placed at the exit pupil of the microscope objective.

Next problem is to state, whether the reconstructed distribution fulfills the assumptions of the above theorem.

The assumption of band-limitedness will be practically satisfied if $f(x)$ is a complex amplitude at the output of an optical system. It is worth noting, moreover, that this assumption is a necessary condition for physical realizability of the derivative $f'(x)$.

The cut-off frequency is either known from the design parameters of the optical systems or it may be measured.

The most difficult problem is to decide whether $f(z)$ and $f'(z)$ have a finite number of zeros in the upper half-plane. As if has been shown in [7] the sufficient condition for $f(z)$ to have a finite number of zeros in the upper half-plane is the following inequality

$$|F(\omega_0)| > |F(-\omega_0)|. \tag{15}$$

The opposite direction of the inequality means that the number of zeros of $f(z)$ is finite but in the lower half-plane. The equality of terms occurring in (15) gives no information about the distribution of zeros. It is easy to notice that the condition (15) assures that $f'(z)$ has also a finite number of zeros in the upper half-plane. Therefore, if we state (by an additional measurement in the frequency plane) that either the inequality (15) is true or an opposite inequality takes place, then this information suffices to determine the distribution of zeros for both $f(z)$ and $f'(z)$.

Unfortunately, if instead of (15) a respective equality occurs, then the information available before the recovery procedure will not allow to distinguish at least two solutions of $f(x)$ and $f^*(x)$ which correspond to phase distribution (4a, b). Then the distribution $F(\omega)$ may be modified in such a way that the equality stops to occur.

This may be achieved in two ways:

a) by illuminating the object in the microscope system under certain known angle [7],

b) by changing the dimensions of the frequency plane.

Of course, a new complex amplitude distribution $f_1(x)$ will be subject to measurement and recovery and $f(x)$ will be determined first thereafter.

To make this result of practical importance it is necessary to find such a method of recovery, which will distinguish the solutions fulfilling the assumptions of our theorem from those which do not satisfy them. Such a method will be soon presented together with computer simulations.

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Условия, достаточные для однозначной реконструкции фазы из распределений напряжённостей, происходящих от комплексной амплитуды и её производной

В работе показано, что знание напряжённостей $|f(x)|^2$ и $\frac{df(x)^2}{dx}$ достаточно для однозначного установления распределения фазы. Достаточно только определить значение частоты среза и установить, является ли напряжённость, происходящая от пространственного спектра, в точке с координатой, равной правой частоте среза, большей (меньшей), чем напряжённости в точке с координатой, равной левой частоте среза.