Optica Applicata, Vol. X, No. 3, 1980

# Generalized treatment of Fourier transforming by lenses*

EUGENIUSZ JAGOSZEWSKI

Institute of Physics, Technical University of Wrocław, Wrocław, Poland.

So far we have found in literature the Fourier transforming properties of thin lenses, only. This paper considers, in general, transforming of properties a thick lens and their influence on the spatial frequencies of Fourier spectrum. Finally, a comparison between the spatial frequencies of spectrum obtained by a thin and by a thick lens is shown.

## Introduction

It is well known that the complex amplitude distribution in the output plane of an elementary optical system can be expressed by the Fourier transform of a distribution in the input plane. The papers so far published were exclusively concerned with the problem of Fourier transforming by thin lenses. An exception is the paper [6] which considers the Fourier transform properties of a plano-convex thick lens. The present work is devoted to the case of double-convex thick lens.

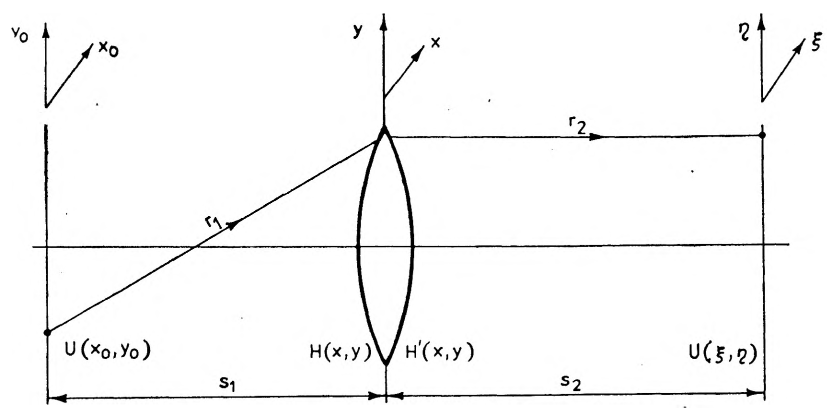


Fig. 1. An elementary optical system to obtain the Fourier transform

Consider a perfectly corrected lens system shown in fig. 1, where the lens is infinitely thin. Using the Huygens–Fresnel principle with assumption that the distances of $s_1$ and $s_2$ are much greater than the maximum diameter of the lens aperture and the diameter of the respective regions in the $(x_0, y_0)$ and $(\xi, \eta)$ planes, the com-

* This work was carried on under the Research Project M. R. I. 5.

plex amplitude at points $(x, y)$ and $(\xi, \eta)$ can be written as

$$H(x, y) = \frac{1}{i\lambda} \int_{X_0} \int_{Y_0} U_0(x_0, y_0) \frac{\exp(ikr_1)}{r_1} dx_0 \; dy_0 \tag{1}$$

and

$$U(\xi, \eta) = \frac{1}{i\lambda} \int_{X} \int_{Y} H'(x, y) \frac{\exp(ikr_2)}{r_2} dx dy, \tag{2}$$

respectively, where the obliquity factor is approximated by one. Since functions $U_0(x_0, y_0)$ and $H'(x, y)$ are equal to zero outside the apertures $(X_0, Y_0)$ and $(X, Y)$, the superposition integrals (1), (2) can be written with infinite limits. Assuming the Fresnel approximation [1], we can write the superposition integrals as a convolution of $U_0(x_0, y_0)$ (or of $H'(x, y)$) with the weighting function

$$H(x, y) = A_1 \left\{ U_0(x, y) \otimes \exp\left[ i \frac{k}{2s_1} (x^2+y^2) \right] \right\}, \tag{3}$$

$$U(\xi, \eta) = A_2 \left\{ H'(\xi, \eta) \otimes \exp\left[ i \frac{k}{2s_2} (\xi^2\eta+\eta^2) \right] \right\}, \tag{4}$$

where

$$A_1 = \frac{\exp(iks_1)}{is_1\lambda}, \quad A_2 = \frac{\exp(iks_2)}{is_2\lambda}.$$

It is well known, too [1] that a plane wave normally incident on a lens forms a spherical wave behind the lens. Thus we observe the field transformation as a quadratic approximation to a spherical wave. If the focal length of the lens is positive (converging lens), then the field distribution behind the lens is written as

$$H'(x, y) = H(x, y) \exp\left[ -i \frac{k}{2f_0} (x^2+y^2) \right] \exp(iknd), \tag{5}$$

where $H(x, y)$ is the field distribution in front of the lens. Of course, we assume that the slope angles of incident rays are small enough to make the formula (5), valid for thin lenses, applicable to the thick lenses, as well. By combining the expressions (3) (4), and (5), we obtain the following form of field distribution in the output plane

$$\begin{aligned} U(\xi, \eta) = & \frac{1}{s_1 s_2 \lambda^2} \iiiint_{-\infty}^{\infty} U_0(x_0, y_0) \\ & \times \exp\left\{ i \frac{k}{2s_1} \left[ (x-x_0)^2+(y-y_0)^2 \right] \right\} \exp\left[ -i \frac{k}{2f_0} (x^2+y^2) \right] \\ & \times \exp\left\{ i \frac{k}{2s_2} \left[ (\xi-x)^2+(\eta-y)^2 \right] \right\} dx_0 dy_0 dx dy, \end{aligned} \tag{6}$$

where the constant phase factor has been dropped. By integrating the field distribution in the lens plane $(x, y)$ we obtain

$$U(\xi,\eta) = \frac{\exp\left[i\frac{k}{2s_2}(\xi^2+\eta^2)\right]}{\lambda s_1 s_2\left(\frac{1}{s_1}+\frac{1}{s_2}-\frac{1}{f_0}\right)}\cdot\iint\limits_{-\infty}^{\infty} U_0(x_0,y_0)\exp\left[i\frac{k}{2s_1}(x_0^2+y_0^2)\right]$$

$$\times\exp\left\{-i\frac{k}{2\left(\frac{1}{s_1}+\frac{1}{s_2}-\frac{1}{f_0}\right)}\left[\left(\frac{x_0}{s_1}+\frac{\xi}{s_2}\right)^2+\left(\frac{y_0}{s_1}+\frac{\eta}{s_2}\right)^2\right]\right\}dx_0dy_0. \tag{7}$$

Expression (7) represents the field distribution in the output plane at any distances of the input and of the output planes from the lens. Two cases are very important. First is when the output plane is the image of the input plane, and the second case when the complex amplitude distribution in the output plane determines the Fourier transform of the input distribution. We are interested in the second case which takes place when $s_1 = s_2 = f_0$. Then the resulting amplitude distribution is given by

$$U(\xi,\eta) = \frac{1}{f_0\lambda}\iint\limits_{-\infty}^{\infty} U_0(x_0,y_0)\exp\left[-i\frac{k}{f_0}(x_0\xi+y_0\eta)\right]dx_0dy_0. \tag{8}$$

Thus the field distribution in the back focal plane is seen as the two dimensional Fourier transform of the amplitude distribution in the front focal plane with the folowing spatial frequencies

$$u = \frac{\xi}{f_0\lambda}, \quad v = \frac{\eta}{f_0\lambda}. \tag{9}$$

## Modification of the focal length of a lens and phase transformation

Consider a thick converging lens with two spherical surfaces surrounded by air. Assuming a perfectly corrected lens the object-image correspondence can be studied with good accuracy if the two principal planes which characterized the investigated lens are introduced. Certainly, the principal planes of a thin lens are covered with each other. Figure 2 shows the object principal plane $P$ and the image principal plane $P'$ in a thick lens. Its focal length can be determine from the equation

$$\frac{1}{f} = \frac{1}{f_0} - \frac{1}{f_d}, \tag{10}$$

where

$$\frac{1}{f_0} = (n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) > 0$$

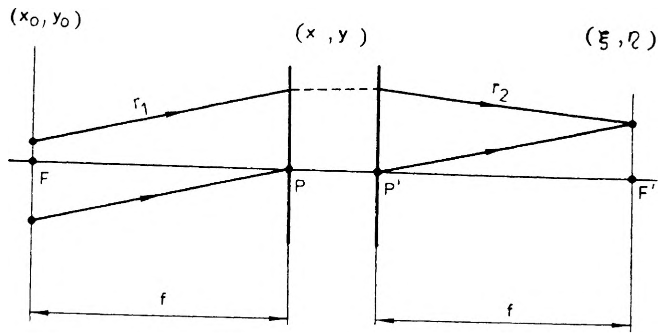


Fig. 2. Ray-tracing of the diffracted light between two focal planes $F$, $F'$ of a thick lens represented by its principal planes $P$, $P'$

s the optical power of a thin lens with the curvatures $1/R_1$, $1/R_2$, respectively, and

$$\frac{1}{f_d} = -\frac{(n-1)^2 d}{nR_1 R_2} > 0$$

represents an additional optical power introduced by the thickness $d$ of the considered lens. Thus we see that the focal length of a thick converging lens is longer than the focal length of thin lens with the same curvatures. Therefore we can analyse the lens considered as an optical system containing two thin lenses (one converging and the other diverging) very close to each other. If the radii of the first lens are $R_1$ and $R_2$, respectively, the radii of the second lens are defined by

$$R_3 = \frac{R_1 R_2}{d} < 0, \tag{11}$$

$$R_4 = \frac{nR_1 R_2}{d} < 0, \tag{12}$$

where $n$ is the index of refraction of the lens medium, and $d$ is the thickness of the thick lens measured along its optical axis.

If the image principal plane of the considered lens overlaps the principal plane of the thin lens having the same curvatures and the same index of refraction, then the suitable image focal planes are shifted relatively to each other. The distance between the two image focal planes is determined by

$$\Delta f = \frac{f_0^2}{f_d - f_0}, \tag{13}$$

and is shown in fig. 3. Remember that the phase delay suffered by the wave at coordinates $(x, y)$ in passing through the lens depends on the focal length only [1], [6]. Assuming a constant value of the axial thickness ($d$ = constant) of the lens, the multiplicative phase transformation may be represented by

$$\exp(iknd)\exp\left[-i\frac{k(f_d - f)}{2f_d f_0}(x^2+y^2)\right], \tag{14}$$

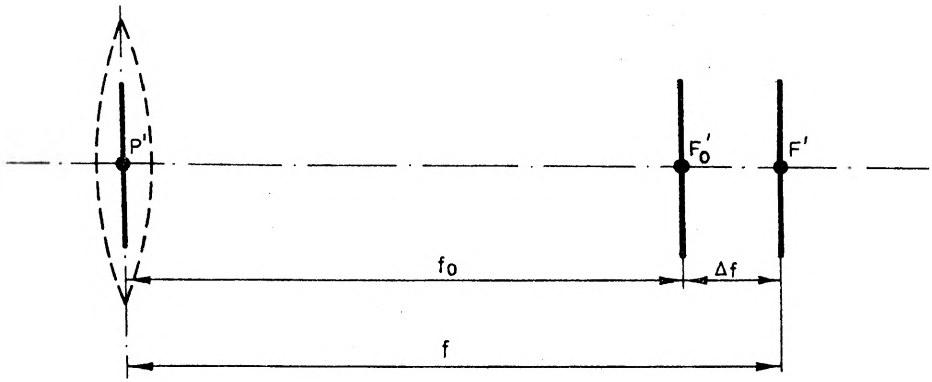


Fig. 3. The distance between the two image focal planes of the thin and of the thick lens, respectively

thus the light distribution behind the lens described by expression (5) takes the form

$$H'(x,y) = H(x,y)\exp(iknd)\cdot\exp\left[-i\frac{k}{2}\left(\frac{1}{f_0}-\frac{1}{f_d}\right)(x^2+y^2)\right]. \tag{15}$$

The equation (15) shows clearly the dependence of the phase transformation on the lens thickness. We see that in a special case when the thickness $d$ tends to zero, the general form of expression (15) takes the form (5), or

$$H'(x,y) = H(x,y)\exp\left[-i\frac{k}{2f_0}(x^2+y^2)\right].$$

## Spatial frequencies of Fourier spectrum realized by a thick lens

It is clear that a lens is a useful device for performing the Fourier transform upon a light field distribution in its input focal plane. An optical system can be constructed by arranging a sequence of lenses which forms a succession of Fourier transform planes. Let us then consider the complex amplitude distribution as a signal $U_0(x_0, y_0)$ in the input focal plane of a thick lens which can be imagine as an optical system of two thin lenses. This optical system can be represented by its object and image principal planes, and is shown in fig. 2. Inserting into the expression (2) or (4) the function $H'(x, y)$ described by (15), instead fo function (5), and putting the distance $s_1 = s_2 = f$, we obtain the field distribution in the image focal plane of the considered lens. If the constant phase factor is neglected, the field distribution becomes

$$U(\xi,\eta) = \frac{f_d-f_0}{\lambda f_d f_0}\int\int_{-\infty}^{\infty} U_0(x_0,y_0)\cdot\exp\left[-i\frac{k}{f_0}\left(1-\frac{f_0}{f_d}\right)(x\xi+y\eta)\right]dx_0\,dy_0.$$

We see that the complex amplitude distribution in the output focal plane of a thick lens is also determined by the Fourier transform of distribution in the input focal plane. There is moreover a difference refering to the description of the Fourier transform of the input function in a thin lens system and in a thick lens system. The point is that in a thick lens system the intermediate functions $H(x, y)$ and $H'(x, y)$ are determined in two planes, i.e. in the two principal planes which are in a certain distance from each other, while in a thin lens system these functions are defined in one plane passing through the middle of the lens. Obviously, this plane is simultaneously the object and the image principal plane. Therefore, in both the cases of thick and thin lenses the optical path lengths $r_1$ and $r_2$ in the object and image regions, respectively, have the same form (figs. 1 and 2). Take notice of

$$\begin{aligned} r_1 &= [f^2+(x-x_0)^2+(y-y_0)^2]^{1/2}, \\ r_2 &= [f^2+(x-\xi)^2+(y-\eta)^2]^{1/2}. \end{aligned} \tag{17}$$

Expression (16) describes the Fourier transform of the input field distribution, evaluated at radian spatial frequencies

$$\begin{aligned} \omega_x &= \frac{2\pi}{\lambda f_0}\left(1-\frac{f_0}{f_d}\right)\xi, \\ \omega_y &= \frac{2\pi}{\lambda f_0}\left(1-\frac{f_0}{f_d}\right)\eta. \end{aligned} \tag{18}$$

Denoting by $\omega_{0x} = \dfrac{2\pi\xi}{\lambda f_0}, \omega_{0y} = \dfrac{2\pi\eta}{\lambda f_0}$ the radian spatial frequencies performed by the thin lens, we get the following relations:

$$\begin{aligned} \omega_x &= \omega_{0x}\left(1-\frac{f_0}{f_d}\right), \\ \omega_y &= \omega_{0y}\left(1-\frac{f_0}{f_d}\right). \end{aligned} \tag{19}$$

This means that the spatial frequencies obtained by a thick lens have smaller values than the appropriate frequencies realized by a thin lens. Suitable relation is plotted in fig. 4. The relative error of the spatial frequency produced by a thickness of lens is a linear function of the thickness and expressed in the following form

$$\frac{\omega_0-\omega}{\omega_0} = \frac{f_0}{f_d}. \tag{20}$$

The graph of the error function presented in fig. 4 has been plotted for three different values of focal length assuming that the curvature of $R_1^{-1} = R_2^{-1}$, and the index of refraction $n = 1.5$.

## Conclusion

From the above considerations it follows that the spatial frequencies of Fourier spectrum obtained by a thick lens depend on the lens thickness. It is shown that a thick converging lens can be represented as a system containing two thin lenses.

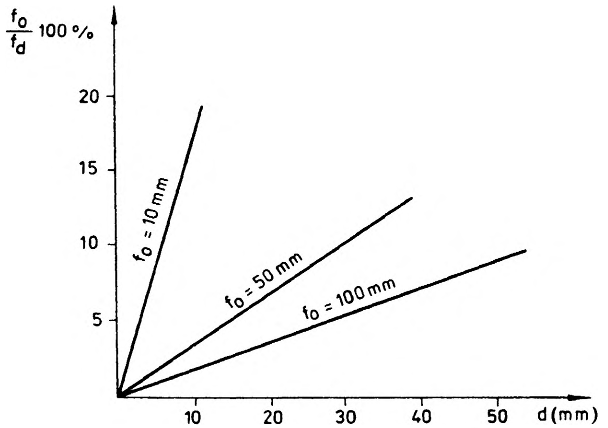


Fig. 4. Relative error of the spatial frequency expressed in percent as a function of the thickness of a lens

Therefore the phase transformation induced by thick lens differs from that produced by the thin lens, since the focal length is a function of thickness of the first lens.

## References

[1] GOODMAN J. W., *Introduction to Fourier Optics*, McGraw-Hill Book Co., New York 1968.
[2] CATHEY W. T., Jr., J. Opt. Soc. Am. **56** (1966), 1015.
[3] LEE T. C., GOSSEN D., Appl. Opt. **10** (1971), 961.
[4] BORN M., WOLF E., *Principles of Optics*, Pergamon Press Inc., New York 1964.
[5] RHODES J. E., Jr., Am. J. Phys. **21** (1953), 337.
[6] JAGOSZEWSKI E., *Recent in Optical Physics*, Proceedings of the ICO-10, 695, Prague 1976.

*Received, October 10, 1979,*
*in revised form, December 12, 1979*

## Обобщенная трактовка трансформации Фурье, осуществляемой линзами

Преобразование Фурье, применяемое в решении основных вопросов современной оптики, осуществляется с помощью фокусирующих оптических элементов. Публикуемые в литературе работы из этой области касаются трансформации Фурье, полученной в системе с бесконечно тонкой линзой.

В настоящей работе описаны транформирующие свойства толстой линзы и их влияние на пространственные частоты спектра Фурье. Заметные различия пространственных частот спектра, осуществляемого с помощью толстой линзы, а также спектра, реализованного с помощью тонкой линзы, обусловлены разностью оптической силой этих линз. Наконец, на примере трёх линз с различными фокусными расстояниями показаны относительные разности пространственных частот, выраженные в процентах, в зависимости от толщины этих линз.