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On the possibility of optical performing of non-integer order derivatives

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The concept of the non-integer order derivative $D^r f(x)$

By virtue of the derivative theorem for Fourier-transformable functions $f(x)$ which says that if the function $f(x)$ has a Fourier-transform $F(u)$, then the derivative $D^r f(x)$ has the Fourier-transform $(2\pi ui)^n F(u)$, for $n = 0, 1, 2$, it is possible to introduce the concept of a derivative of non-integer order. Let a differential operator D^r for positive non-integer r be called a derivative of non-integer order of $f(x)$ if $D^r f(x)$ has the Fourier-transform of the form $(2\pi iu)^r F(u)$. Hence, it is obvious that by using the inverse Fourier transform the following representation of $D^r f(x)$ is obtained:

$$D^r f(x) = \int_{-\infty}^{+\infty} (2\pi ui)^r F(u) \exp(2\pi uix) du. \tag{1}$$

It is easy to note that by admitting the negative non-integer values of r an integral of non-integer order may also be defined in a similar way.

Example

As an illustration of the above concept let us calculate a special case of non-integer order derivative in the form of the fractional order derivative defined by taking $0 \leqslant r \leqslant 1$. Let the function $f(x)$ be the triangle function $\wedge(x)$, defined as

$$\wedge(x) = \begin{cases} 1+x & \text{for } -1 \leqslant x \leqslant 0, \\ 1-x & \text{for } 0 \leqslant x \leqslant 1, \\ 0 & \text{for otherwise,} \end{cases}$$

which has the Fourier transform

$$F(u) = \frac{\sin^2 \pi u}{\pi^2 u^2}.$$

Then, according to (1),

$$D^r f(x) = \int_{-\infty}^{+\infty} (2\pi ui)^r \frac{\sin^2 \pi u}{\pi^2 u^2} \exp(2\pi uxi) du. \tag{2}$$

After some rearrangement we obtain four integrals

$$D^r f(x) = \frac{(2\pi)^r}{\pi^2}\left\{\cos\frac{\pi r}{2}\left[\int\limits_0^\infty u^{r-2}\sin\pi u\sin\pi u(1+2x)du + \int\limits_0^\infty u^{r-2}\sin\pi u\sin\pi u(1-2x)\,du\right] + \right.$$

$$\left. +\sin\frac{\pi r}{2}\left[\int\limits_0^\infty u^{r-2}\sin\pi u\cos\pi u(1+2x)du - \int\limits_0^\infty u^{r-2}\sin\pi u\cos\pi u(1-2x)du\right]\right\} \tag{3}$$

By evaluating the above integrals we can obtain four different expressions valid in the respective ranges of argument of the $\wedge(x)$ function, i.e.:

$$\begin{aligned} &D^r f(x) = 0, && \text{for } x \leqslant -1,\\ &D^r f(x) = C(r)(1+x)^{1-r}, && \text{for } -1 < x \leqslant 0,\\ &D^r f(x) = C(r)[(1+x)^{1-r}-2x^{1-r}], && \text{for } 0 < x \leqslant 1,\\ &D^r f(x) = C(r)[(1+x)^{1-r}-2x^{1-r}+(x-1)^{1-r}], && \text{for } x < 1, \end{aligned} \tag{4}$$

where

$$C(r) = \frac{2\Gamma(r)\sin\pi r}{\pi(1-r)}.$$

The graphs of these functions are shown in figs. 1–3. As it may be seen in the limit cases r $= 0$ or $r = 1$ the function $D^r f(x)$ tends, respectively to the function $D^0 f(x) = \wedge(x)$ or $D^1 f(x)$ which is in accordance with the classical concept of the zero and first derivatives of the differentiated function.

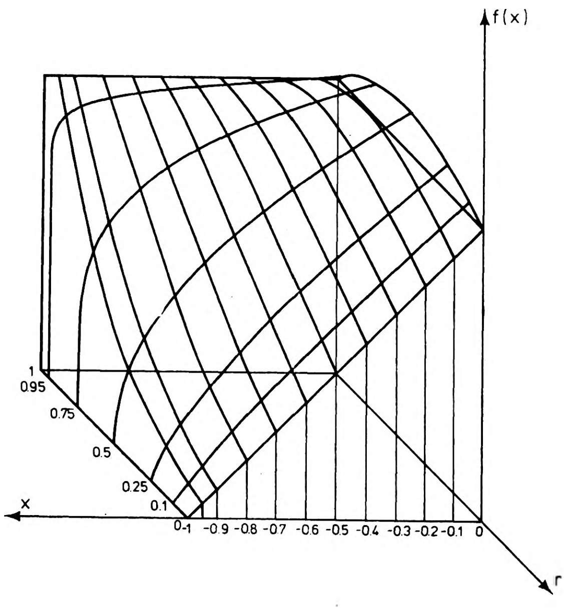

Fig. 1

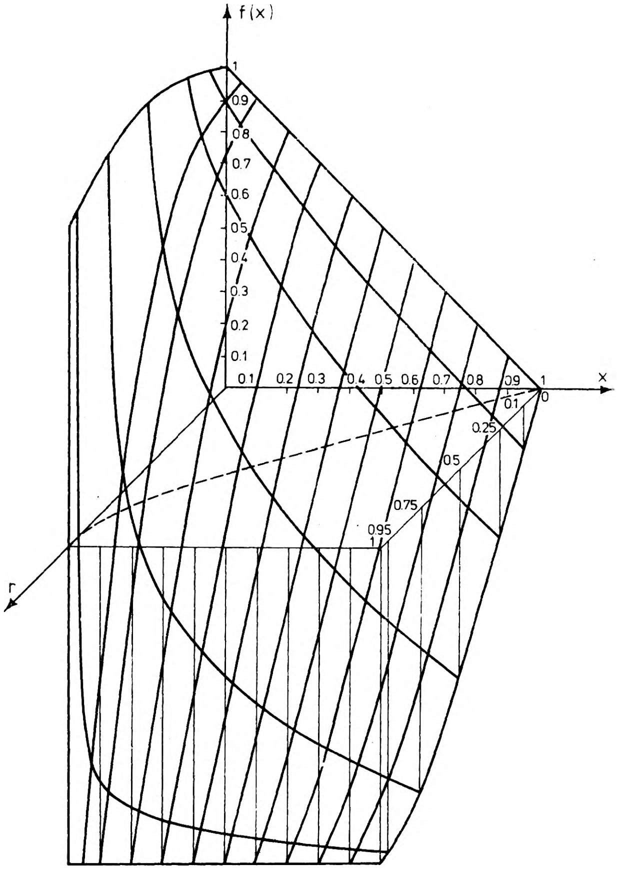

Fig. 2

Optical realization of D^r-operator

The so defined operation of fractional order differentiation may be realized with the help of the respective optical filtering. For this purpose the filter of structure $(2\pi ui)^r$ must be produced. When applying the classical holographic method for filter production, the task is to calculate the intensity distribution in the interference pattern obtained, for instance, by interfering a reference plane wave $A\exp\left(2\pi i\,\dfrac{\sin\Theta}{\lambda}\,u\right)$ falling under the incidence angle Θ on the pattern plane with the wave of complex amplitude $(2\pi iu)^r$, i.e.

$$J = \left|A\exp\left(2\pi ui\frac{\sin\Theta}{\lambda}u\right)+(2\pi ui)^r\right|^2. \tag{5}$$

After substituting $a = \sin\Theta/\lambda$ we have for $u > 0$:

$$J = A^2+(2\pi u)^{2r}+2(2\pi u)A\cos\left(2\pi au+r\frac{\pi}{2}\right). \tag{6}$$

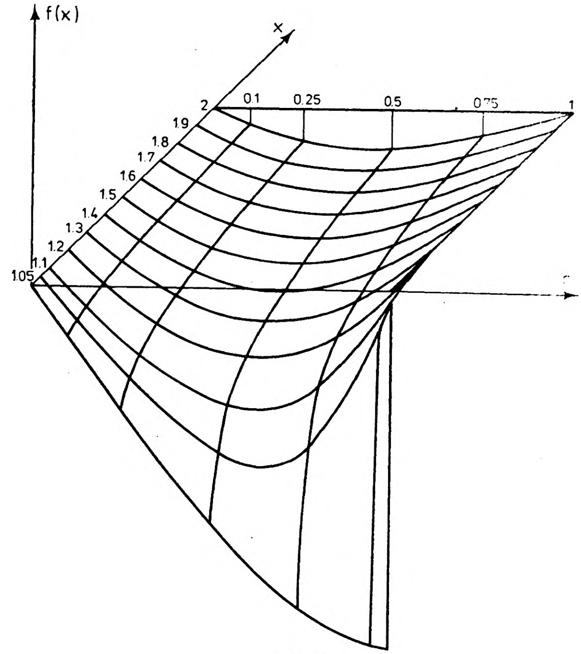

Fig. 3

It may be easily shown, that for $u < 0$ the intensity J will be of mirror-reflection symmetry with respect to the case $u > 0$. By replacing the first two terms in (6) by one constant value such that the intensity at the edge of the filter be equal to zero we may rewrite the formula (6) in the form

$$J = C\left[u^r_{\max} + u^r \cos\left(2\pi au + r\frac{\pi}{2}\right)\right] = C_1\left[1 + \frac{u^r}{u^r_{\max}} \cos\left(2\pi au + r\frac{\pi}{2}\right)\right], \tag{7}$$

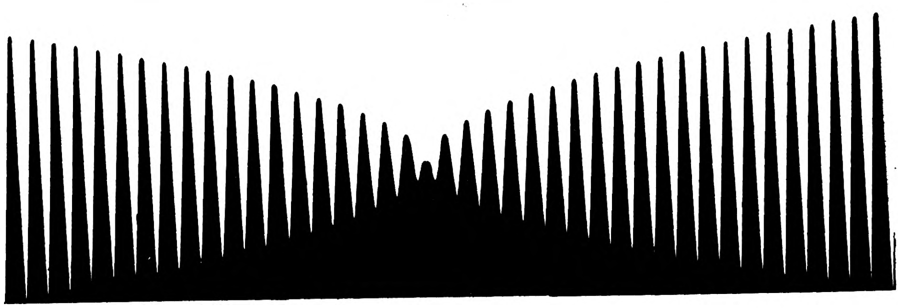

Fig. 4

where C_1 and C_2 are some constant, which may be suitably chosen to fit the exposure conditions during filter recording. The structure represented by the expression (7) should be recorded on a material of linear response, whereby the intensity changes are coded in the respective changes in optical density of the filter. Figure 4 shows the cross-section of the optical density in a filter calculated for $r = 1/2$ by using the above method.

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